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(Partial Differential Equations and Analytical and Numerical Approaches)

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Module 1

Partial Differential Equation

Topics & Contents:

* Origin of Partial Differential Equations (PDEs)
* Linear and Nonlinear PDEs of First Order
* Lagrange's Equations
* Charpit's Method
* Cauchy's Method of Characteristics
* Solution of linear PDEs of higher order with constant coefficients
* Equations reducible to linear PDEs with constant coefficients

Motivations and Context : The aim of this is to introduce and motivate partial differential equations (PDE). The section also places the scope of studies in APM346 within the vast universe of mathematics. A partial differential equation (PDE) is an gather involving partial derivatives. This is not so informative so let's break it down a bit. The mathematical formulations of many problems in science and engineering reduce to study of first-order PDEs. For instance, the study of first-order PDEs arise in gas flow problems, traffic flow problems, phenomenon of shock waves, the motion of wave fronts, It is therefore essential to study the theory of first-order PDEs and the nature their solutions to analyze the related physical problems. we shall study first-order linear, and nonlinear PDEs and methods of solving these equations. An important method of characteristics is explained for these equations in which solving PDE reduces to solving an ODE system along a characteristics curve. Further, the Charpit's method for nonlinear first-order PDEs are discussed. Partial Differential Equations arise when physical laws involve functions that depend on more than one independent variable, such as space and time. They are obtained by eliminating arbitrary constants or arbitrary functions from equations.

A Partial Differential Equation (PDE) involves an unknown function $z = f(x, y)$and its partial derivatives with respect to independent variables xand y.

- Standard Notation:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial xy}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Order of a PDE: The order of the highest derivative term in the equation is called the order of the PDE.

- Formation:

1. By Eliminating Arbitrary Constants: For $f(x, y, z, a, b) = 0$, differentiate partially with respect to x and y to eliminate a and b.

Example. Form a partial differential equation from $x^2 + y^2 + (z - c)^2 = a^2$.

Solution. $x^2 + y^2 + (z - c)^2 = a^2 \quad \ldots(1)$

(1) contains two arbitrary constants a and c.

Differentiating (1) partially w. r. t. x we get

$2x + 2(z - c)\,(\partial z/\partial x) = 0$

or $x + (z - c)\,p = 0 \quad \ldots(2)$

Differentiating (1) partially w. r. t. y we get

$2y + 2(z - c)\,(\partial z/\partial y) = 0$

or $y + (z - c)\,q = 0 \quad \ldots(3)$

Let us eliminate c from (2) and (3)

From (2): $(z - c) = -x/p$

Putting this value of $z - c$ in (3), we get

$y - (x/p)\,q = 0$

or $yp - xq = 0$.

2**. By Eliminating Arbitrary Functions: For $\Phi(u, v) = 0$, differentiate partially with respect to xand yto eliminate the arbitrary function Φ.**

Example. Form the partial differential equation from $z = f(x^2 - y^2)$

Solution. $z = f(x^2 - y^2) \quad \ldots(1)$

Differentiating (1) w. r. t. x and y:

$p = \partial z/\partial x = f'(x^2 - y^2) \cdot 2x \quad \ldots(2)$

$q = \partial z/\partial y = f'(x^2 - y^2) \cdot (-2y) \quad \ldots(3)$

Dividing (2) by (3) we get:

$p/q = -x/y$ or $py = -qx$

or $yp + xq = 0$.

Linear & Non-Linear First-Order PDEs:

- Lagrange's Quasi-Linear PDE: An equation of the form:

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

- Non-Linear First-Order PDE: An equation involving p and q in non-linear forms, classified into standard types or solved using Charpit's Method.

Derivations & Analytical Methods:

Method 1: Lagrange's Method for Quasi-Linear First-Order PDEs

Derivation:

Let the quasi-linear PDE be:

$$\mathrm{P(x,y,z)p + Q(x,y,z)q = R(x,y,z)} \quad \text{--- (1)}$$

Let $\mathrm{u(x,y,z) = c_1}$ and $\mathrm{v(x,y,z) = c_2}$ be two independent solutions of the associated auxiliary system. Taking total differentials:

$$\mathrm{du} = \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \mathrm{u}}{\partial \mathrm{z}}\mathrm{dz} = 0$$

$$\mathrm{dv} = \frac{\partial \mathrm{v}}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \mathrm{v}}{\partial \mathrm{z}}\mathrm{dz} = 0$$

Since zis a function of xand y, substituting $\mathrm{dz = p\,dx + q\,dy}$:

$$\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{p}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)\mathrm{dx} + \left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{q}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)\mathrm{dy} = 0$$

Equating coefficients of dxand dyto zero gives:

$$\mathrm{p} = -\frac{\partial \mathrm{u}/\,\partial \mathrm{x}}{\partial \mathrm{u}/\,\partial \mathrm{z}}, \quad \mathrm{q} = -\frac{\partial \mathrm{u}/\,\partial \mathrm{y}}{\partial \mathrm{u}/\,\partial \mathrm{z}}$$

Substituting pand qinto (1):

$$\mathrm{P}\left(-\frac{\partial \mathrm{u}/\,\partial \mathrm{x}}{\partial \mathrm{u}/\,\partial \mathrm{z}}\right) + \mathrm{Q}\left(-\frac{\partial \mathrm{u}/\,\partial \mathrm{y}}{\partial \mathrm{u}/\,\partial \mathrm{z}}\right) = \mathrm{R}$$

$$\Rightarrow \mathrm{P}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{Q}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{R}\frac{\partial \mathrm{u}}{\partial \mathrm{z}} = 0$$

This yields the Lagrange Subsidiary/Auxiliary Equations:

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Example:

Solve $x(y-z)p + y(z-x)q = z(x-y)$.

Solution:

Auxiliary Equations:

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Using Multipliers $(1,1,1)$:

$$\frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{0}$$

$$\Longrightarrow dx + dy + dz = 0 \Longrightarrow x + y + z = c_1$$

Using Multipliers $(1/x, 1/y, 1/z)$:

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Longrightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Longrightarrow \ln x + \ln y + \ln z = \ln c_2 \Longrightarrow xyz = c_2$$

General Solution: $\Phi(x+y+z, xyz) = 0$.

Method 2: Cauchy's Characteristic Method:

Derivation: Consider a first-order quasilinear partial differential equation of the form:

$$a\,(\partial u / \partial x) + b\,(\partial u / \partial y) = f(x,y) + K\,u,\ \ u(0,y) = h(y) \quad \text{--- (1)}$$

Here coefficients a, b, and f depend on x, y, and u, but not on the derivatives of u.

Let $u(x,y)$ be the solution of equation (1). Total derivative:

$$du = (\partial u / \partial x)\,dx + (\partial u / \partial y)\,dy \quad \text{(Total derivative)}$$

$$\Rightarrow\ dx\,(\partial u / \partial x) + dy\,(\partial u / \partial y) = du \quad --- \ (2)$$

From (1) and (2), we get the Auxiliary Equations (A.E.):

$$dx / a = dy / b = du / [f(x,y) + K u] \quad --- (3)$$

Taking the first two fractions from equation (3):

$$dx / a = dy / b \Rightarrow b\,dx - a\,dy = 0$$

$$\text{On integrating: } b\,x - a\,y = c \quad --- (4)$$

Taking the first and third fractions from equation (3):

$$dx / a = du / [f(x,y) + K u]$$

Substituting y = (b x - c) / a from equation (4):

$$dx / a = du / [f(x, (bx - c)/a) + K u]$$

$$\Rightarrow (du / dx) - (K / a)\, u = (1 / a)\, f(x, (bx - c)/a) \quad --- (5)$$

which is an ODE in u. The complete solution of (5) is:

$$u = G(x, c) + c_1 \quad --- (6)$$

where $c_1 = g(c)$ and replacing $c = bx - ay$ from equation (4):

$$\Rightarrow u = G(x, c) + g(c) \Rightarrow u(x, y) = G(x, bx - ay) + g(bx - ay).$$

Example:

Solve : $(\partial u / \partial x) + (\partial u / \partial y) = x + y, \quad u(x, 0) = 0$

Solution: Here $(\partial u / \partial x) + (\partial u / \partial y) = x + y \quad --- (1)$

Here A.E. is:

$$dx / 1 = dy / 1 = du / (x + y) \quad --- (2)$$

Taking first two fractions:

$$dx = dy \Rightarrow dx - dy = 0$$

On integrating: $x - y = c_1 \quad --- (3)$

Taking second and third fractions:

$$\Rightarrow dy = du/(x + y)$$

$$\Rightarrow dy = du / (2y + c_1) \quad \text{from (3)}$$

$$\Rightarrow du = (2y + c_1)\, dy$$

On integrating:

$$u(x,y) = y^2 + c_1 y + c_2$$

$$\text{Let } c_2 = g(c_1) \text{ then:}$$

$$\Rightarrow u(x,y) = y^2 + (x - y)y + g(x - y)$$

$$\Rightarrow u(x,y) = x\,y + g(x - y) \quad ---\ (5)$$

where g is an arbitrary function.

Condition: $u(x, 0) = 0$

$$\text{From (5): } u(x,0) = 0 + g(x) \Rightarrow 0 = g(x)$$

$$\therefore g(x - y) = 0 \quad ---\ (6)$$

Using (6) in (5):

$$u(x,y) = x\,y$$

$$\text{General solution: } u(x,y) = x\,y.$$

Solve the following First-Order Partial Differential Equations:

1. $u_x - y\,u = 0, \text{ with condition } u(0,y) = 1$
2. $u_x + u_y = u,$

   $\text{with condition } u(x,0) = 1 + e^x$ Ans. $u(x,y) = e^x + e^y$

Method 3: Charpit's Method for Non-Linear First-Order PDEs:

Derivation: Consider a general non-linear first-order PDE:

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

We seek an additional relation $g(x, y, z, p, q) = 0$ --- (2) such that pand qcan be solved from (1) and (2) to render $dz = p\,dx + q\,dy$ integrable.

Differentiating (1) and (2) partially with respect to xand y:

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial g}{\partial x} + p\frac{\partial g}{\partial z} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = 0$$

Eliminating $\frac{\partial p}{\partial x}$ and using the compatibility conditions $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$, we obtain Charpit's Auxiliary Equations:

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

1. **Example:**

Example. Solve: $px + qy = pq$

Solution. $f(x, y, z, p, q) = 0$ is $px + qy - pq = 0 \ \ldots(1)$

$\partial f/\partial x = p, \ \partial f/\partial y = q, \ \partial f/\partial z = 0, \ \partial f/\partial p = x - q, \ \partial f/\partial q = y - p$

Charpit's equations are:

$$dx/-(x - q) = dy/-(y - p) = dz/[-p(x - q) - q(y - p)] = dp/p = dq/q = d\phi/0$$

We have to choose the simplest integral involving p and q:

$dp/p = dq/q \Rightarrow \log p = \log q + \log a \Rightarrow p = a q$

Putting for p in the given equation (1), we get:

$q(a x + y) = a q^2 \Rightarrow q = (y + a x)/a$

$\therefore \ p = a q = y + a x$

Now, $dz = p\,dx + q\,dy \ \ldots(2)$

Putting for p and q in (2), we get:

$dz = (y + a x)\,dx + [(y + a x)/a]\,dy$

$a\,dz = (y + a x)\,dx + (y + a x)\,dy$

$a\,dz = (y + a x)(a\,dx + dy)$

Integrating:

$a z = (y + a x)^2/2 + b$

General solution: $a z = \frac{1}{2}(y + a x)^2 + b$.

Higher-order linear PDEs with constant coefficient:

$$a_0\left(\partial^n z/\partial x^n\right) + a_1\left(\partial^n z/\partial x^{n-1}\partial y\right) + \ldots + a_n\left(\partial^n z/\partial y^n\right) = F(x, y) \ \ldots(1)$$

is called a homogeneous linear partial differential equation of nth order with constant Coeffients .

It is called homogeneous partial differential equation.

$$(a_0 D^n + a_1 D^{n-1} D' + \ldots + a_n D'^n)\, z = F(x, y)$$

or

$f(D, D')z = F(x, y)$

RULES FOR FINDING THE COMPLEMENTARY FUNCTION (C. F.):

Consider the equation

$a_0 (\partial^2 z / \partial x^2) + a_1 (\partial^2 z / \partial x \partial y) + a_2 (\partial^2 z / \partial y^2) = 0$

or $(a_0 D^2 + a_1 D D' + a_2 D'^2) z = 0$

1st step: Put $D = m$ and $D' = 1$ to obtain the auxiliary equation:

$a_0 m^2 + a_1 m + a_2 = 0$

2nd step: Solve the auxiliary equation.

Example . Solve: $(D^3 - 4 D^2 D' + 3 D D'^2) z = 0$

Solution. Given $(D^3 - 4 D^2 D' + 3 D D'^2) z = 0$. Put $D = m, D' = 1$.

Its auxiliary equation is:

$m^3 - 4 m^2 + 3 m = 0$ or $m (m^2 - 4 m + 3) = 0$

$m (m - 1)(m - 3) = 0 \Rightarrow m = 0, 1, 3$

The required solution is: $z = f_1(y) + f_2(y + x) + f_3(y + 3 x)$

Example . Solve: $\partial^2 z / \partial x^2 - 4 \partial^2 z/(\partial x \partial y) + 4 \partial^2 z / \partial y^2 = 0$

Solution. $(D^2 - 4 D D' + 4 D'^2) z = 0$. Put $D = m, D' = 1$.

Its auxiliary equation is:

$m^2 - 4 m + 4 = 0$ or $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

The required solution is: $z = f_1(y + 2 x) + x f_2(y + 2 x)$.

EXERCISE :

Solve the following equations:

1. $\partial^2 z / \partial x^2 + 4 \partial^2 z/(\partial x \partial y) - 5 \partial^2 z / \partial y^2 = 0$ z $= f_1(y + x) + f_2(y - 5x$

2. $\frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 0$ z $= f_1(2y - x) + f_2(y - 2x)$.

RULES FOR FINDING THE PARTICULAR INTEGRAL (P. I.):

Given partial differential equation: $f(D, D') Z = F(x, y)$

P. I. $= [1 / f(D, D')] F(x, y)$

(i) When $F(x, y) = e^{(ax+by)}$

P. I. $= [1 / f(D, D')]\, e^{(ax+by)} = e^{(ax+by)} / f(a, b)$ [Put $D = a, D' = b$]

(ii) When $F(x, y) = \sin(a\,x + b\,y)$ or $\cos(a\,x + b\,y)$

P. I. $= [1 / f(D^2, D\,D', D'^2)] \sin(a\,x + b\,y)$ or $\cos(a\,x + b\,y)$

Put $D^2 = -a^2, D\,D' = -a\,b, D'^2 = -b^2$:

P. I. $= [\sin(a\,x + b\,y)$ or $\cos(a\,x + b\,y)] / f(-a^2, -a\,b, -b^2)$

(iii) When $F(x, y) = x^m y^n$

P. I. $= [1 / f(D, D')]\, x^m y^n = [f(D, D')]^{-1}\, x^m y^n$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D′ and operate on $x^m y^n$ term by term.

(iv) When $F(x, y) =$ Any function

P. I. $= [1 / (D - m\,D')]\, F(x, y)$

$= \int F(x, c - m\,x)\, dx$ where c is replaced by $y + m\,x$ after integration.

Case 1. When R. H. S. $= e^{(ax+by)}$

Example

Solve: $\partial^3 z / \partial x^3 - 3\,\partial^3 z / (\partial x^2\, \partial y) + 4\,\partial^3 z / \partial y^3 = e^{(x+2y)}$

Solution. Given equation in symbolic form is:

$(D^3 - 3\,D^2\,D' + 4\,D'^3)\, z = e^{(x+2y)}$

Its A. E. is $m^3 - 3\,m^2 + 4 = 0$, whence $m = -1, 2, 2$.

C. F. $= f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x)$

P. I. $= [1 / (D^3 - 3\,D^2\,D' + 4\,D'^3)]\, e^{(x+2y)}$

Put $D = 1, D' = 2$:

P. I. $= e^{(x+2y)} / (1 - 6 + 32) = e^{(x+2y)} / 27$

Hence complete solution is:

$z = f_1(y - x) + f_2(y + 2x) + x\, f_3(y + 2x) + (1/27) e^{(x+2y)}$

EXERCISE :

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{(x+2y)}$

   Ans. $z = f_1(y + x) + f_2(y - x) - \frac{1}{3} e^{(x+2y)}$

2. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\, \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$ Ans. z

   $= f_1(y + 2x) + f_2(y + 3x) + \frac{1}{2} e^{x+y}$

Case 2. When R. H. S. = sin(a x + b y) or cos(a x + b y)

Example.

Solve: $\partial^3 z/\partial x^3 - 4\,\partial^3 z/(\partial x^2\,\partial y) + 4\,\partial^3 z/(\partial x\,\partial y^2) = 2\sin(3x + 2y)$.

Solution. Putting $\partial/\partial x = D, \partial/\partial y = D'$:

$$(D^3 - 4D^2D' + 4DD'^2)z = 2sin(3x + 2y)$$

$$A.E.\, m(m-2)^2 = 0 \Rightarrow m = 0,2,2.$$

$$C.F. = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$$

$$P.I. = 2 \cdot [1/D(D^2 - 4DD' + 4D'^2)]sin(3x + 2y)$$

$$Put D^2 = -9, DD' = -6, D'^2 = -4:$$

$$P.I. = 2 \cdot \left[1/D\left(-9 - 4(-6) + 4(-4)\right)\right]sin(3x + 2y)$$

$$= (-2/D)sin(3x + 2y)$$

$$= -⅔[-cos(3x + 2y)]$$

$$= ⅔cos(3x + 2y)$$

General solution : $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + ⅔cos(3x + 2y)$

EXERCISE :

1. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x.$ z $= f_1(y + x) + x f_2(y + x) - \sin x$

2. $(D^2 - D D')z = \cos x \cos 2.$ $z = f_1(y) + f_2(y + x) + ½\cos(x + 2y) - \left(\frac{1}{6}\right)\cos(x - 2y)$.

Case 3. When R. H. S. = $x^m y^n$

Example. Find the general integral of: $\partial^2 z/\partial x^2 + 3\partial^2 z/(\partial x \partial y) + 2\partial^2 z/\partial y^2 = x + y$.

Solution. $(D^2 + 3DD' + 2D'^2)z = x + y$

A. E. : $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2.$

C. F. $= f_1(y - x) + f_2(y - 2x)$

$P.I. = [1/(D^2 + 3DD' + 2D'^2)](x + y)$

$= (1/D^2)[1 + (3D'/D) + (2D'^2/D^2)]^{-1}(x + y)$

$= (1/D^2)[1 - (3D'/D) + \ldots](x + y)$

$= (1/D^2)[x + y - (3/D)(1)]$

$= (1/D^2)[y - 2x]$

$= ½x^2y - (x^3/3)$

Hence complete solution:

$z = f_1(y - x) + f_2(y - 2x) + ½\,x^2y - ⅓\,x^3.$

EXERCISE

Solve the following partial differential equations:

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y.$ $\quad z = f_1(y-x) + f_2(y+x) + (x^3/6) - (x^2y/2)$

2. $\frac{\partial^2 z}{\partial x^2} + 3\left(\frac{\partial^2 z}{\partial x\,\partial y}\right) + 2\left(\frac{\partial^2 z}{\partial y^2}\right) = 12xy.$ $z = f_1(y-x) + f_2(y-2x) + 2x^3y - (3x^4/2)$

2. Case IV. When R. H. S. = Any Function

Example

Solve $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y - 6(\partial^2 z/\partial y^2) = ycosx.$

Solution:

In operator form: $(D^2 + DD' - 6D'^2)z = ycosx$

Auxiliary Equation: $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = 2, -3$

C. F. $= f_1(y + 2x) + f_2(y - 3x)$

P. I. $= [1/(D - 2D')(D + 3D')]ycosx$

$Puty = c + 3x$:

$= [1/(D - 2D')]\int (c + 3x)cosxdx = [1/(D - 2D')][(c + 3x)sinx + 3cosx]$

$= [1/(D - 2D')][ysinx + 3cosx]$

$Puty = c - 2x$:

$= \int [(c - 2x)sinx + 3cosx]dx = -(c - 2x)cosx - 2sinx + 3sinx$

$= -ycosx + sinx$

Hence, the complete solution is:

$z = f_1(y + 2x) + f_2(y - 3x) + sinx - ycosx$

EXERCISE

Solve the following equations:

1. $(D - D')(D + 2D')z = (y + 1)e^x.$ $\quad z = f_1(y + x) + f_2(y - 2x) + y\,e^x.$
2. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \tan^3 x \tan y - \tan x \tan^3 y.$ $\quad z = f_1(y + x) + f_2(y - x) + (1/2)\tan x \tan y.$

Question Bank

1. Form the PDE by eliminating aand bfrom $z = (x + a)(y + b)$. **Ans :** $z = p\,q$
2. Define quasi-linear, semilinear, and non-linear partial differential equations.
3. Write down Charpit's auxiliary equations for $f(x, y, z, p, q) = 0$.
   Ans : $dx/f_p = dy/f_q = dz/(pf_p + qf_q) = -dp/(f_x + pf_z) = -dq/(f_y + qf_z)$
4. State the condition for the equation $P\,dx + Q\,dy + R\,dz = 0$to be integrable.
   $AnsP\partial Q/\partial z - \partial R/\partial y + Q\partial R/\partial x - \partial P/\partial z + R\partial P/\partial y - \partial Q/\partial x = 0$
5. Solve $p + q = 1$. ***Ans*** $z = ax + (1 - a)y + c$
6. Find the complete integral of $p^2 + q^2 = 1$. **Ans** $z = ax + \sqrt{(1 - a^2)}y + c$
7. Solve $z = px + qy + pq$(Clairaut's form). **Ans** : $z = ax + by + ab$
8. Find the CF of $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.
   Ans. $CF = f_1(y + x) + xf_2(y + x) + x^2f_3(y + x)$
9. Find the PI of $(D^2 - 2DD' + D'^2)z = e^{x+y}$. **Ans** $PI = (1/2)x^2e^{(x+y)}$
10. Define the Cauchy Problem for first-order PDEs.
11. Solve $p\tan x + q\tan y = \tan z$. **Ans** $sinz = sinx \cdot \varphi(sinx/siny)$
12. Find the general solution of $D(D - 2D' - 1)z = 0$. **Ans** $z = f_1(y) + e^xf_2(y + 2x)$
13. Transform $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 0$ into constant coefficient form.
    Ans $With\ x = e^u and\ y = e^v$: $[D_u(D_u - 1) - D_v(D_v - 1)]z = 0$
14. What are characteristic curves of a first-order linear PDE?
15. Find the complete integral of $p = z + qx$. **Ans** : $z = ae^x + bxe^x orz = ce^x - ax$

1. Derive Cauchy's Method of Characteristics for solving a general non-linear first-order PDE $f(x, y, z, p, q) = 0$.

Ans: $System\ of characteristic\ ODEs: dx/ds = f_p, dy/ds = f_q, dz/ds = pf_p + qf_q, dp/ds = -(f_x + pf_z), dq/ds = -(f_y + qf_z)$

2. Solve using Charpit's Method: $(p^2 + q^2)y = qz$. **Ans** : $z^2 = a^2y^2 + (x + b)^2$
3. Find the complete integral and singular integral of $z = px + qy + p^2 + q^2$.
   Ans: $CompleteIntegral: z = ax + by + a^2 + b^2 \bullet SingularIntegral: 4z = -(x^2 + y^2)$
4. Solve Lagrange's equation: $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.
   Ans: $\varphi\big(x + y + z, (x - y)/(y - z)\big) = 0$
5. Solve $(y + z)p + (z + x)q = x + y$.
   Ans: $\varphi\big((x + y + z)(x - y)^2, (x - y)/(y - z)\big) = 0$
6. Using Charpit's Method, solve $px + qy = pq$. **Ans** : $z = ax + by + ab$.
7. $u_x + u_y = u$, with condition $u(x, 0) = 1 + e^x$, $u(x, y) = e^x + e^y$.
   Ans: $u(x, y) = e^y + e^{(x+y)}$
8. Solve the reducible non-homogeneous equation: $(D^2 - D'^2 + d - D')z = e^{2x+3y} + xy$.
   Ans: : $z = f_1(y + x) + e^{(-x)}f_2(y - x) + (1/6)e^{(2x+3y)} - xy - x + y + 1$
9. Find the general solution of $(D^2 - 5DD' + 6D'^2)z = y \sin x$.
   Ans: $z = f_1(y + 2x) + f_2(y + 3x) - ysinx - 5cosx$
10. Solve $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^2$using transformation variables $x = e^u, y = e^v$. **Ans:** $z = f_1(yx^2) + lnx \cdot f_2(yx^2) + (1/6)x^3y^2$

Module 2

Variable Separable Solutions to Linear PDEs

Topics & Contents:

* Variable separable solutions to Linear Partial Differential Equations
* Classification of linear partial differential equations of second order
* Method of separation of variables
* Solution of the wave and heat conduction equations up to two dimensions
* Laplace equation in two dimensions

Classification of Second-Order Linear Partial Differential Equations: The general second-order linear partial differential equation in two independent variables $(x, y)$is given by:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y)$$

where coefficients $A, B, C, D, E,$ Fare real constants or functions of xand y. The classification depends strictly on the discriminant $B^2 - 4AC$:

Discriminant Condition	Type of PDE	Canonical / Physical Prototype	Physical Example
$B^2 - 4AC < 0$	Elliptic	Laplace Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Steady-state temperature, Electrostatics
$B^2 - 4AC = 0$	Parabolic	1D Heat Conduction Equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	Transient heat flow, Diffusion processes

Discriminant Condition	Type of PDE	Canonical / Physical Prototype	Physical Example
$B^2 - 4AC > 0$	Hyperbolic	1D Wave Equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	Vibrating strings, Acoustic waves

1.**The Method of Separation of Variables:** This method assumes that the solution to a linear homogeneous partial differential equation can be factored into a product of functions, each depending on a single independent variable.

For a function $u(x, t)$:

$$u(x, t) = X(x) \cdot T(t)$$

Substituting this product into the PDE transforms the partial differential equation into a set of ordinary differential equations (ODEs), linked by a constant known as the separation constant (k or $-\lambda^2$).

2. Comprehensive Method Derivations: Derivation Complete Solution of One-Dimensional Wave Equation:

The 1D Wave Equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Step 1: Assume a product solution

$$u(x, t) = X(x)T(t)$$

Differentiating partially with respect to xand t:

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t), \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

Step 2: Separate variables Substitute into (1):

$$X(x)T''(t) = c^2 X''(x)T(t)$$

Divide both sides by $c^2X(x)T(t)$:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = K \quad \text{(Separation Constant)}$$

This yields two coupled Ordinary Differential Equations:

1. $X''(x) - KX(x) = 0$
2. $T''(t) - Kc^2T(t) = 0$

Step 3: Analyze possible values of K

- Case I: When $K = 0$

$$X'' = 0 \Longrightarrow X(x) = C_1x + C_2$$

$$T'' = 0 \Longrightarrow T(t) = C_3t + C_4$$

$$u(x,t) = (C_1x + C_2)(C_3t + C_4)$$

(Non-periodic; does not satisfy boundary conditions of fixed endpoints).

- Case II: When $K = p^2 > 0$

$$X'' - p^2X = 0 \Longrightarrow X(x) = C_5e^{px} + C_6e^{-px}$$

$$T'' - p^2c^2T = 0 \Longrightarrow T(t) = C_7e^{pct} + C_8e^{-pct}$$

$$u(x,t) = (C_5e^{px} + C_6e^{-px})(C_7e^{pct} + C_8e^{-pct})$$

(Exponential growth/decay; physically invalid for periodic wave vibration).

- Case III: When $K = -p^2 < 0$

$$X'' + p^2X = 0 \Longrightarrow X(x) = C_9\cos(px) + C_{10}\sin(px)$$

$$T'' + p^2c^2T = 0 \Longrightarrow T(t) = C_{11}\cos(pct) + C_{12}\sin(pct)$$

$$u(x,t) = (C_9\cos(px) + C_{10}\sin(px))(C_{11}\cos(pct) + C_{12}\sin(pct))$$

Conclusion: Case III is the only physically correct solution representing wave motion.

Derivation : Complete Solution of One-Dimensional Heat Conduction Equation

The 1D Heat Conduction Equation is:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Step 1: Variable separation Let $u(x,t) = X(x)T(t)$. Substituting partial derivatives:

$$XT' = c^2 X''T \Longrightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = K$$

Step 2: Solve the coupled ODEs

1. $X'' - KX = 0$
2. $T' - Kc^2 T = 0$

Setting $K = -p^2$to ensure thermal decay over time $t \to \infty$:

$$X'' + p^2 X = 0 \Longrightarrow X(x) = C_1 \cos(px) + C_2 \sin(px)$$

$$T' + p^2 c^2 T = 0 \Longrightarrow \frac{dT}{T} = -p^2 c^2 dt \Longrightarrow T(t) = C_3 e^{-p^2 c^2 t}$$

Step 3: General Solution

$$u(x,t) = (C_1 \cos p\,x + C_2 \sin p\,x) e^{-p^2 c^2 t}$$

Derivation: Solution of Two-Dimensional Laplace Equation in Cartesian Coordinates:

Laplace's equation in 2D steady-state conditions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Assuming $u(x,y) = X(x)Y(y)$:

$$X''Y + XY'' = 0 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} = K$$

Assuming separation constant $K = -p^2$:

1. $X'' + p^2 X = 0 \Longrightarrow X(x) = C_1 \cos p\,x + C_2 \sin p\,x$
2. $Y'' - p^2 Y = 0 \Longrightarrow Y(y) = C_3 \cosh p\,y + C_4 \sinh p\,y$

The resulting standard steady-state solution is:

$$u(x,y) = (C_1 \cos p\,x + C_2 \sin p\,x)(C_3 \cosh p\,y + C_4 \sinh p\,y)$$

3. Step-by-Step Solved Examples:

Example (Wave Equation)

Problem: A tightly stretched string of length lwith fixed ends $x = 0$and $x =$ lis initially at rest in its equilibrium position. If it is set vibrating by giving each of its points a velocity $v_0 \sin\left(\frac{\pi x}{l}\right)$, find the displacement $u(x, t)$at any time t.

Solution:

- Boundary Conditions (BCs):
    1. $u(0, t) = 0, \quad \forall t \geq 0$
    2. $u(l, t) = 0, \quad \forall t \geq 0$
- Initial Conditions (ICs): 3. $u(x, 0) = 0$ 4. $\frac{\partial u}{\partial t}\Big|_{t=0} = v_0 \sin\left(\frac{\pi x}{l}\right)$

The general periodic solution is:

$$u(x, t) = (C_1 \cos p\, x + C_2 \sin p\, x)(C_3 \cos p\, ct + C_4 \sin p\, ct)$$

Applying BC (1): $u(0, t) = 0 \Longrightarrow C_1(C_3 \cos p\, ct + C_4 \sin p\, ct) = 0 \Longrightarrow C_1 = 0.$

$$u(x, t) = C_2 \sin(px)\,(C_3 \cos p\, ct + C_4 \sin p\, ct)$$

Applying BC (2): $u(l, t) = 0 \Longrightarrow C_2 \sin(pl)\,(\ldots) = 0 \Longrightarrow \sin(pl) = 0 \Longrightarrow pl = n\pi \Longrightarrow p = \frac{n\pi}{l}.$

$$u(x, t) = \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right]$$

Applying IC (3): $u(x, 0) = 0 \Longrightarrow A_n \sin\left(\frac{n\pi x}{l}\right) = 0 \Longrightarrow A_n = 0.$

$$u(x, t) = B_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$$

Differentiating with respect to t:

$$\frac{\partial u}{\partial t} = B_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

Applying IC (4) at $t = 0$:

$$\sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = v_0 \sin\left(\frac{\pi x}{l}\right)$$

Comparing terms for $n = 1$:

$$B_1 \left(\frac{\pi c}{l}\right) = v_0 \Longrightarrow B_1 = \frac{v_0 l}{\pi c}$$

For $n \neq 1$, $B_n = 0$.

Substituting B_1 back into the solution yields:

$$u(x,t) = \frac{v_0 l}{\pi c} \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi c t}{l}\right)$$

Example (Heat Equation):

Problem: A rod of length L has its ends A and B kept at 0°C. The initial temperature distribution in the rod is $u(x,0) = 50x(L-x)$. Find the temperature distribution $u(x,t)$ at any subsequent time t.

Solution: The general solution for the 1D heat equation is:

$$u(x,t) = (C_1 \cos p\,x + C_2 \sin p\,x)e^{-p^2c^2t}$$

Applying BCs $u(0,t) = 0$ and $u(L,t) = 0$:

$$C_1 = 0 \quad \text{and} \quad p = \frac{n\pi}{L}$$

By superposition:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2c^2t}{L^2}}$$

Applying IC $u(x,0) = 50x(L-x)$:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = 50x(L-x)$$

Using Fourier Sine Series coefficient formula:

$$b_n = \frac{2}{L}\int_0^L 50\,x(L-x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{100}{L}\int_0^L (Lx - x^2) \sin\left(\frac{n\pi x}{L}\right) dx$$

Integrating by parts:

$$b_n = \frac{100}{L}\left[(Lx - x^2)\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right) - (L-2x)\left(-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right) + (-2)\left(\frac{L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right)\right]_0^L$$

$$b_n = \frac{100}{L}\left[-\frac{2L^3}{n^3\pi^3}\cos(n\pi) + \frac{2L^3}{n^3\pi^3}\right] = \frac{200L^2}{n^3\pi^3}[1-(-1)^n]$$

$$b_n = \begin{cases} \frac{400L^2}{n^3\pi^3} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Final solution:

$$u(x,t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{400L^2}{n^3\pi^3} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2c^2t}{L^2}}$$

Question Bank

1. Classify the partial differential equation: $3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$. Ans

$$y\ parabolic(Since B^2 - 4AC = 4^2 - 4(3)(1) = 4 > 0)$$

2. Write down the discriminant expression used to classify second-order linear PDEs .Ans
3. Classify the two-dimensional Laplace equation.
4. State the physical significance of the separation constant $K < 0$in the 1D wave equation.
5. Write all three possible mathematical solutions for the one-dimensional heat equation obtained via variable separation.
6. What are the boundary conditions for a vibrating string fixed at both ends $x = 0$and $x = L$?
7. Write the transmission line equations (Telegrapher's equations) in terms of voltage Vand current I.
8. Convert the radio-frequency transmission line equation into a high-frequency wave equation format.
9. State the steady-state conditions for a 1D heat conduction bar.
10. Define the fundamental frequency of vibration for a stretched string of length L.
11. Classify the equation: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.
12. Write the 2D Laplace equation in polar coordinates (r, θ).
13. Which form of solution is suitable for solving Laplace's equation when $u(x,y) \to 0$as $y \to \infty$?
14. Give an example of a parabolic partial differential equation occurring in engineering physics.

15. What physical quantity does c^2represent in the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$?

16. Define steady-state temperature distribution in a thin rectangular plate.

17. Write the boundary conditions for an insulated end at $x = 0$in a heat conduction problem.

18. Classify $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$for region $x > 0, y > 0$.

    Ans. $Elliptic(SinceB^2 - 4AC = 0 - 4(x)(y) = -4xy < 0 for x > 0, y > 0)$

19. State the principle of superposition for solutions of homogeneous linear PDEs.

20. Write down the 2D heat equation in steady state.

1. Classify the general second-order PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$and reduce the equation $y^2u_{xx} - x^2u_{yy} = 0$to its canonical form for $x > 0, y > 0$.

   Ans. Canonical Form $(y^2u_x x - x^2u_y y = 0\ is\ Hyperbolic)$: $\partial^2 u/(\partial\xi\partial\eta) = (1/4\xi\eta)[\eta(\partial u/\partial\xi) + \xi(\partial u/\partial\eta)] where \xi = y^2 + x^2, \eta = y^2 - x^2$

2. A tightly stretched string with fixed end points $x = 0$and $x = l$is initially in its position of equilibrium. If it is set vibrating by giving each point a velocity $v = kx(l - x)$, find the displacement $u(x, t)$at any distance xand time t.

   Ans. $u(x,t) = \left(8kl^3/(\pi^4 c)\right)\sum_{n=1,3,5,\ldots}^{\infty}(1/n^4)sin(n\pi x/l)sin(n\pi ct/l)$

3. Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$subject to the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$for all $t > 0$, and the initial condition $u(x, 0) = \sin\left(\frac{3\pi x}{l}\right) + 2\sin\left(\frac{5\pi x}{l}\right)$.

   Ans. $u(x,t) = sin(3\pi x/l)e^{(-9\pi^2 c^2 t/l^2)} + 2sin(5\pi x/l)e^{(-25\pi^2 c^2 t/l^2)}$

4. A uniform bar of length Lhas its ends maintained at 0°C. The initial temperature throughout the bar is given by $u(x, 0) = 100\sin\left(\frac{\pi x}{L}\right)$. Find $u(x, t)$.

   Ans. $u(x,t) = 100sin(\pi x/L)\ e^{(-\pi^2 c^2 t/L^2)}$

5. Derive the equations of high-frequency transmission lines (Telegrapher's equations) and prove that both voltage Vand current Isatisfy the wave equation when resistance Rand conductance Gare negligible.

6. A rectangular plate with insulated surfaces is bounded by $x = 0, x = a, y = 0, y = b$. The temperatures along the edges are $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$, and $u(x, 0) = f(x)$. Find the steady-state temperature $u(x, y)$in the plate.

Ans. $u(x,y) = \sum_{n=1}^{\infty} A_n sin(n\pi x/a) sinh(n\pi(b-y)/a) where A_n = [2/(a sinh(n\pi b/a))] \int_0^a f(x) sin(n\pi x/a) dx$

7. Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$in a square plate of side 10 cmsubjected to boundary conditions: $u(0, y) = 0$, $u(10, y) = 0$, $u(x, 10) = 0$, and $u(x, 0) = 100$.

Ans. $u(x,y) = (400/\pi) \sum_{n=1,3,5,\ldots}^{\infty} [sin(n\pi x/10) sinh(n\pi(10-y)/10)]/[n sinh(n\pi)]$

8. A string is stretched and fastened to two points at $x = 0$and $x = l$. Motion is started by displacing the string into the form $u(x, 0) = A \sin\left(\frac{\pi x}{l}\right)$from which it is released at time $t = 0$. Find the displacement $u(x, t)$.

Ans. $u(x,t) = A sin(\pi x/l) cos(\pi c t/l)$

9. Find the steady-state temperature distribution in a thin rectangular plate bounded by $x = 0, x = l, y = 0, y = \infty$, where the edges $x = 0$and $x = l$are kept at 0°C, the edge $y = \infty$is maintained at 0°C, and the edge $y = 0$is kept at constant temperature U_0.

Ans. $u(x,y) = (4U_0/\pi) \sum_{n=1,3,5,\ldots}^{\infty} (1/n) sin(n\pi x/l) e^{(-n\pi y/l)}$

10. Solve the 1D heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$subject to boundary conditions $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$, $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$, and initial condition $u(x, 0) = x$.

Ans. $u(x,t) = (L/2) - (4L/\pi^2) \sum_{n=1,3,5,\ldots}^{\infty} (1/n^2) cos(n\pi x/L) e^{(-\alpha(n\pi/L)^2 t)}$

11. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}$for a string of length $l = \pi$with fixed ends, initial velocity zero, and initial displacement $u(x, 0) = \frac{1}{3} \sin x - \frac{1}{6} \sin 3x$.

Ans. $u(x,t) = (1/3) sinx cos(2t) - (1/6) sin(3x) cos(6t)$

12. An infinitely long plane uniform plate is bounded by two parallel edges $x = 0$and $x = \pi$and an end at $y = 0$. The sides $x = 0$and $x = \pi$are kept at 0°C. The end $y = 0$is kept at temperature $u(x, 0) = kx(\pi - x)$. Find the steady-state temperature distribution.

Ans. $u(x,y) = (4/\pi)\sum_{n=1,3,5,\ldots}^{\infty}(\pi n)sin(nx)e^{(-ny)}\left[or 2\sum_{n=1}^{\infty}\left((-1)^{(n+1)}+1\right)/ nsin(nx)e^{(-ny)}\right]$

13. Derive the two-dimensional heat equation in Cartesian coordinates under non-steady state conditions and reduce it to Laplace's equation for steady state.

14. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$subject to $u(0,y)=0$, $u(l,y)=0$, $u(x,0)=0$, and $u(x,a)=\sin\left(\frac{n\pi x}{l}\right)$.

Ans. $u(x,y) = sin(n\pi x/l)[sinh(n\pi y/l)/sinh(n\pi a/l)]$

15. A string of length L is released from rest from the initial position $u(x,0)=\frac{4h}{L^2}x(L-x)$. Derive the expression for the displacement $u(x,t)$.

Ans. $u(x,t) = (8h/\pi^3)\sum_{n=1,3,5,\ldots}^{\infty}(1/n^3)sin(n\pi x/L)cos(n\pi ct/L)$

16. Derive the separation of variables form for the 2D Laplace equation in polar coordinates $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$.

Ans. $(1/R)d/dr[rdR/dr] = \lambda^2, (1/\Theta)d^2\Theta/d\theta^2 = -\lambda^2 u(r,\theta) = \left(Ar^{\lambda}+ Br^{(-\lambda)}\right)(Ccos\lambda\theta + Dsin\lambda\theta)$

Module-3

Laplace and Fourier Transforms

Topics & Contents:

- Review of Fourier Series
- Fourier Transform
- Fourier sine and Fourier cosine transforms and their properties
- Review on Laplace Transform and Inverse Laplace Transforms

Periodic Functions: A function $f(t)$ is periodic if its value repeats at equal intervals on the horizontal axis:

$$f(t) = f(t + T) = f(t + 2T) = \cdots$$

Here, T is the period of $f(t)$. For example, $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi)$, so $\sin x$ is periodic with period 2π.

Fourier Series: A non-sinusoidal periodic function can be expanded into a series of sines and cosines:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

In summation notation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n x + \sum_{n=1}^{\infty} b_n \sin n x$$

- a_0: Constant term (d.c. component in electrical systems)
- a_1, b_1: Fundamental frequency components
- a_n, b_n: Harmonics and Fourier coefficients

Dirichlet's Conditions: For $f(x)$ to be expandable in a Fourier series over $(-\pi, \pi)$, it must satisfy:

1. $f(x)$ is single-valued and bounded.
2. It has a finite number of maxima, minima, and discontinuities.
3. $f(x + 2\pi) = f(x)$ outside $[-\pi, \pi]$.

At continuous points, the series converges to $f(x)$. At points of discontinuity, it converges to:

$$\frac{1}{2}[f(x + 0) + f(x - 0)]$$

Advantages of Fourier Series: Discontinuous functions can be represented (unlike Taylor series).

- Useful for expanding functions periodic only outside a closed interval.
- Expands oscillating functions into all oscillation modes.
- Term-by-term integration is generally valid.

Useful Standard Integrals: $\int_0^{2\pi} \sin n\,x\,dx = 0$

- $\int_0^{2\pi} \cos n\,x\,dx = 0$
- $\int_0^{2\pi} \sin^2 n\,x\,dx = \pi$
- $\int_0^{2\pi} \cos^2 n\,x\,dx = \pi$
- $\int_0^{2\pi} \sin n\,x \cdot \sin m\,x\,dx = 0 \quad (n \neq m)$
- $\int_0^{2\pi} \cos n\,x \cdot \cos m\,x\,dx = 0 \quad (n \neq m)$
- $\int_0^{2\pi} \sin n\,x \cdot \cos m\,x\,dx = 0$
- $\sin n\,\pi = 0, \quad \cos n\,\pi = (-1)^n \quad (n \in \mathbb{Z})$
- $\int u\,v\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Determination of Fourier Coefficients (Euler's Formulae):

$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots a_n \cos nx + \ldots\ldots\ldots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \ldots\ldots$ (1)

1. **To find** a_0**:** Integrate f(x) from 0 to 2π:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

2. **To find** a_n**:** Multiply f(x) by $\cos n\,x$ and integrate from 0 to 2π:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos n\,x\,dx$$

3. **To find** b_n**:** Multiply f(x) by $\sin n\,x$ and integrate from 0 to 2π:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin n\,x\,dx$$

Example. Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of f(x). Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

Solution. Let $x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$ (1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$
$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx$$
$$= \frac{1}{\pi}\left[(x + x^2)\frac{\sin nx}{n} - (2x + 1)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[(2\pi + 1)\frac{\cos n\pi}{n^2} - (-2\pi + 1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\sin nx\,dx$$
$$= \frac{1}{\pi}\left[(x + x^2)\left(-\frac{\cos nx}{n}\right) - (2x + 1)\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[-(\pi + \pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\cdot 0 + (-\pi + \pi^2)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3}\right]$$
$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get:

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$
$$- 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \quad \ldots(2)$$

Put $x = \pi$ in (2), $\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$ …… (3)

Put $x = -\pi$ in (2), $-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$ ……. (4)

Adding (3) and (4):

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$
$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textbf{Ans.}$$

Example . Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$
$$f(x + 2\pi) = f(x)$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$ ... (1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} x \, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x \cos nx \, dx = \frac{1}{\pi}\left[x \cdot \frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \quad \text{when n is odd} \quad = 0 \quad \text{when n is even.}$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x \sin nx \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{(-1)^n}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ in (1), we get:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right] \quad \textbf{Ans.}$$

Exercise :

1. Find a Fourier series to represent $f(x) = \pi - x$ for $0 < x < 2\pi$.
   Ans. $2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{n}\sin nx + \cdots\right]$

2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that:

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

   Ans. $-\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$

3. Find a Fourier series to represent: $f(x) = x \sin x$, for $0 < x < 2\pi$.
   Ans. $-1 + \pi \sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + \cdots\right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$.

Ans. $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2}-\frac{1}{1^2+1}\cos x+\frac{1}{2^2+1}\cos 2x-\frac{1}{3^2+1}\cos 3x+\cdots\right)+\frac{1}{1^2+1}\sin x-\frac{2}{2^2+1}\sin 2x+\frac{3}{3^2+1}\sin 3x\ldots\right]$; $\frac{\pi}{\sinh\pi}=1+2\left[-\frac{1}{2}+\frac{1}{5}-\frac{1}{10}+\cdots\right]$

Discontinuous Functions:

At a point of discontinuity, Fourier series gives the value of f(x) as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

$$\text{At } x = c, f(x) = \frac{1}{2}[f(c-0)+f(c+0)]$$

Example. Find the Fourier series for f(x), if $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

Solution. Let $f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\cdots+a_n\cos nx+\cdots+b_1\sin x+b_2\sin 2x+\cdots+b_n\sin nx+\cdots$ ... (1)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$\text{Then } a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx+\int_{0}^{\pi}x\,dx\right]=\frac{1}{\pi}[-\pi(x)_{-\pi}^{0}+(x^2/2)_0^{\pi}]=\frac{1}{\pi}(-\pi^2+\pi^2/2)=-\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx+\int_{0}^{\pi}x\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\right]=\frac{1}{\pi n^2}(\cos n\pi-1)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx+\int_{0}^{\pi}x\sin nx\,dx\right]=\frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\,\pi) - \frac{\pi}{n}\cos n\,\pi\right] = \frac{1}{n}(1-2\cos n\,\pi)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \quad \ldots(2)$$

Putting $x = 0$ in (2), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right)$ ... (3)

Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$ $\quad \therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ Proved

Example. Find the Fourier series expansion of the periodic function of period 2π, defined by

$$f(x) = x, \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad f(x) = \pi - x, \quad \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi - x)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi - x)\cos n\,x\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin n\,x}{n} - (1)\left(-\frac{\cos n\,x}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin n\,x}{n} - (-1)\left(-\frac{\cos n\,x}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n} - \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\,\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\,\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x \sin n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)\sin n\,x\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} x \sin n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)\sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos n\,x}{n}\right) - (1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_0^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos n\,x}{n}\right) - (-1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{3\sin\frac{n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\,\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right] = \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ we get:

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right] \quad \textbf{Ans.}$$

Example . Find the Fourier series of the function defined as:

$$f(x) = \begin{cases} x + \pi & \text{for } 0 \le x \le \pi \\ -x - \pi & \text{for } -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x - \pi)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x + \pi)\,dx = \frac{1}{\pi}\left(-\frac{x^2}{2} - \pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2} + \pi x\right)_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \pi^2\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \pi\left(\frac{1}{2} - 1\right) + \pi\left(\frac{1}{2} + 1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n\,x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos n\,x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos n\,x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x - \pi)\cos n\,x\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x + \pi)\cos n\,x\,dx$$

$$= \frac{1}{\pi}\left[(-x - \pi)\frac{\sin n\,x}{n} - (-1)\left\{-\frac{\cos n\,x}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x + \pi)\frac{\sin n\,x}{n} - (1)\left\{-\frac{\cos n\,x}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{n^2\pi} \quad \text{if n is odd.}$$

$$= 0 \quad \text{if n is even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin n\,x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin n\,x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin n\,x\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin n\,x\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos n\,x}{n}\right) - (-1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos n\,x}{n}\right) - (1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1-(-1)^n]$$

$$= \frac{4}{n}, \quad \text{if n is odd.}$$

$$= 0, \quad \text{if n is even.}$$

Fourier series is:

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2\,x + \cdots + b_1\sin x + b_2\sin 2\,x + \cdots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3\,x}{3^2} + \cdots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3\,x}{3} + \cdots\right) \quad \textbf{Ans.}$$

Exercise :

1. Find the Fourier series of the function:

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases} \quad \text{where } f(x+2\pi) = f(x).$$

Ans. $\frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3\,x + \frac{1}{5}\sin 5\,x + \frac{1}{7}\sin 7\,x + \cdots\right]$

2. Find the Fourier series for the function:

$$f(x) = \begin{cases} -\frac{\pi}{4} & \text{for } -\pi < x < 0 \\ \frac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0, f(x) = f(x+2\pi)$ for all x.
Deduce that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad \textbf{Ans.} \ \frac{\sin x}{1} + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \frac{\sin 7\,x}{7} + \cdots$$

3. Find the Fourier series of the function:

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$$

4. Obtain a Fourier series to represent the following periodic function:

$$f(x) = 0 \quad \text{when } 0 < x < \pi$$
$$f(x) = 1 \quad \text{when } \pi < x < 2\pi$$

Ans. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

5. Find the Fourier expansion of the function defined in a single period by the relations:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}$$

and from it deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Ans. $\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

6. Find a Fourier series to represent the function:

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x \leq 0 \\ \frac{1}{4}\pi x & \text{for } 0 < x < \pi \end{cases}$$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Ans. $\frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx + \cdots\right]$

7. Find the Fourier series for f(x), if:

$$f(x) = -\pi \quad \text{for } -\pi < x \leq 0$$
$$= x \quad \text{for } 0 < x < \pi$$
$$= -\frac{\pi}{2} \quad \text{for } x = 0$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Ans. $-\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$

8. Obtain a Fourier series to represent the function:

$$f(x) = |x| \quad \text{for } -\pi < x < \pi$$

and hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Ans. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$

9. Expand as a Fourier series, the function f(x)defined as

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < -\frac{\pi}{2} \\ \frac{\pi}{2} & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

$$\text{Ans. } \frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

10. Obtain a Fourier series to represent the function

$$f(x) = |\sin x| \quad \text{for } -\pi < x < \pi$$

$$\left[\text{Hint } f(x) = \begin{cases} -\sin x & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}\right]$$

$$\text{Ans. } \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right]$$

11. An alternating current after passing through a rectifier has the form

$$i = I\sin\theta \quad \text{for } 0 < \theta < \pi$$

$$= 0 \quad \text{for } \pi < \theta < 2\pi$$

Find the Fourier series of the function.

$$\text{Ans. } \frac{I}{\pi} - \frac{2I}{\pi}\left(\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \cdots\right) + \frac{I}{2}\sin\theta$$

12. If $f(x) = 0$for $-\pi < x < 0$

$$= \sin x \text{ for } 0 < x < \pi$$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}$. Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots\infty = \frac{1}{4}(\pi - 2)$

Even Function: A function f(x)is said to be even (or symmetric) function if,

$$f(-x) = f(x)$$

The graph of such a function is symmetrical with respect to y-axis [f(x) axis]. Here y-axis is a mirror for the reflection of the curve.

The area under such a curve from $-\pi$to $\pi$is double the area from 0to π.

$$\therefore \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n x\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos n x\,dx$$

As f(x)and $\cos n x$are both even functions ∴The product of $f(x)\cdot\cos n x$is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n x\,dx = 0$$

As $\sin nx$ is an odd function so $f(x).\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cos terms.

Odd Function: A function f(x)is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$to $\pi$is zero.

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n x\,dx = 0 \quad [f(x)\cdot\cos n x \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n x\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin n x\,dx$$

$[f(x)\cdot\sin n x$ is even function.]

The series of the odd function will contain only sin terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms.

Example . Find the Fourier series expansion of the periodic function of period 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution.

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

This is an even function. $\therefore b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos n x\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos n x\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin n x}{n}\right) - (2x)\left(-\frac{\cos n x}{n^2}\right) + (2)\left(-\frac{\sin n x}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$

Fourier series is $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + a_n\cos nx + \cdots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right]$$

$$\text{or } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12} \quad \text{Proved.}$$

Example . Obtain a Fourier expression for

$$f(x) = x^3 \quad \text{for } -\pi < x < \pi$$

Solution. $f(x) = x^3$ is an odd function.

$$\therefore a_0 = 0 \quad \text{and } a_n = 0$$

$$\cdot\, b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin n x\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin n x\,dx$$

$$\left[\int u\,v = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos n x}{n}\right) - 3x^2\left(-\frac{\sin n x}{n^2}\right) + 6x\left(\frac{\cos n x}{n^3}\right) - 6\left(\frac{\sin n x}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3}\right] = 2 \cdot (-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right] \quad \text{Ans.}$$

Half-Range Series, Period 0 To π:

The given function is defined in the interval $(0, \pi)$and it is immaterial whatever the function may be outside the interval $(0, \pi)$. To get the series of cosines only we assume that f(x)is an even function in the interval $(-\pi, \pi)$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos n x\, dx \quad \text{and } b_n = 0$$

To expand f(x)as a sine series we extend the function in the interval $(-\pi, \pi)$as an odd function.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin n x\, dx \quad \text{and } a_n = 0$$

Example 10. Represent the following function by a Fourier sin series :

$$f(t) = \begin{cases} t, & 0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < t \le \pi \end{cases}$$

Solution. $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin n t\, dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin n t\, dt + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2}\sin n t\, dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos n t}{n}\right) - (1)\left(-\frac{\sin n t}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos n t}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0 + 1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \cdots \quad \text{Ans.}$$

Example . Find the Fourier sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$ where a is constant.

Solution. $\int e^{ax}\sin b\,x\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin b\,x - b\cos b\,x]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin n\,x - n\cos n\,x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\,\pi - n\cos n\,\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\frac{n}{a^2+n^2}[-(-1)^n e^{a\pi} + 1] = \frac{2n}{(a^2+n^2)\pi}[1 - (-1)^n e^{a\pi}]$$

$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \quad b_2 = \frac{2\cdot 2\cdot(1-e^{a\pi})}{(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(2-e^{a\pi})}{a^2+2^2}\sin 2x + \cdots\right] \quad \text{Ans.}$$

Exercise :

1. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

$$\text{Ans. } \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right]$$

2. Find a series of cosine of multiples of xwhich will represent $f(x)$in $(0, \pi)$where

$$f(x) = 0 \quad \text{for } 0 < x < \frac{\pi}{2}$$

$$f(x) = \frac{\pi}{2} \quad \text{for } \frac{\pi}{2} < x < \pi$$

Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$

Ans. $\frac{\pi}{4} - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \cdots$

3. Express $f(x) = x$as a sin series in $0 < x < \pi$.

Ans. $2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$

4. Find the cosine series for $f(x) = \pi - x$in the interval $0 < x < \pi$.

Ans. $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

5. If $f(x) = x, \quad \text{for } 0 < x < \frac{\pi}{2}$

$$= \pi - x, \quad \text{for } \frac{\pi}{2} < x < \pi$$

Show that:

$$\text{(i) } f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right)$$

$$\text{(ii) } f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \cdots\right)$$

6. Obtain the half-range cosine series for $f(x) = x^2$in $0 < x < \pi$.

Ans. $\frac{\pi^2}{3} - \frac{4}{\pi}\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right)$

7. Find (i) sine series and (ii) cosine series for the function

$$f(x) = e^x \quad \text{for } 0 < x < \pi.$$

Ans. (i) $\frac{2}{\pi}\sum_{n=1}^{\infty} n\left[\frac{1-(-1)^n e^{\pi}}{n^2+1}\right]\sin nx$ (ii) $\frac{e^{\pi}-1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{\pi}}{n^2+1}\cos nx$

8. If $f(x) = x + 1$, for $0 < x < \pi$, find its Fourier (i) sine series (ii) cosine series. Hence deduce that

$$\text{(i) } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \quad \text{(ii) } 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Ans. (i) $\frac{2}{\pi}\left[(\pi+2)\sin x-\frac{\pi}{2}\sin 2x+\frac{1}{3}(\pi+2)\sin 3x-\frac{\pi}{4}\sin 4x+\cdots\right]$

(ii) $\frac{\pi}{2}+1-4\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$

9. Find the Fourier series expansion of the function

$$f(x)=\cos(sx),\quad -\pi\le x\le\pi$$

where s is a fraction. Hence, show that $\cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\cdots$

Ans. $\frac{\sin\pi s}{\pi s}+\frac{1}{\pi}\left(\frac{\sin(s\pi+n\pi)}{s+n}+\frac{\sin(s\pi-n\pi)}{s-n}\right)\cos nx$

Change Of Interval and Functions having Arbitrary Period: In electrical engineering problems, the period of the function is not always 2π but Tor $2c$. This period must be converted to the length 2π. The independent variable xis also to be changed proportionally. Let the function f(x)be defined in the interval $(-c,c)$. Now we want to change the function to the period of 2π so that we can use the formulae of a_n, b_n ,$2c$ is the interval for the variable x.

$$\therefore 1 \text{ is the interval for the variable } =\frac{x}{2c}$$

$$\therefore 2\pi \text{ is the invertal for the variable } =\frac{x\cdot 2\pi}{2c}=\frac{\pi x}{c}$$

$$\text{so put } z=\frac{\pi x}{c} \quad \text{or } x=\frac{zc}{\pi}$$

Thus the function f(x)of period 2cis transformed to the function $f\left(\frac{cz}{\pi}\right)$or F(z)of period 2π.

F(z)can be expanded in the Fourier series.

$$F(z)=f\left(\frac{cz}{\pi}\right)=\frac{a_0}{2}+a_1\cos z+a_2\cos 2z+b_1\sin z+b_2\sin 2z+\cdots$$

where $a_0=\frac{1}{\pi}\int_0^{2\pi}F(z)\,dz=\frac{1}{\pi}\int_0^{2\pi}f\left(\frac{cz}{\pi}\right)\,dz$

$$=\frac{1}{c}\int_0^{2c}f(x)\,d\left(\frac{\pi x}{c}\right)=\frac{1}{c}\int_0^{2c}f(x)\,dx \quad \text{put } z=\frac{\pi x}{c}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}F(z)\cos nz\,dz=\frac{1}{\pi}\int_0^{2\pi}f\left(\frac{cz}{\pi}\right)\cos nx\,dz \quad \text{Put } z=\frac{\pi x}{c}$$

$$=\frac{1}{\pi}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,d\left(\frac{\pi x}{c}\right)=\frac{1}{c}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,dx.$$

$$\text{Similarly } b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Cor. Half range series [Interval $(0, c)$] Cosine series:

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \cdots + a_n\cos\frac{n\pi x}{c} + \cdots$$

$$\text{where } a_0 = \frac{2}{c}\int_0^c f(x)\,dx, \quad a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Sine series.

$$f(x) = b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \cdots + b_n\sin\frac{n\pi x}{c} + \cdots$$

$$\text{where } b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

Example . A periodic function of period 4 is defined as

$$f(x) = |x|, \quad -2 < x < 2.$$

Find its Fourier series expansion.

Solution. $f(x) = |x| \quad -2 < x < 2$

$$f(x) = x \quad 0 < x < 2$$

$$= -x \quad -2 < x < 0$$

$$a_0 = \frac{1}{c}\int_{-c}^c f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^0 (-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^0 = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^0 (-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+ \frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^0$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{8}{n^2\pi^2} \quad \text{if n is odd.}$$

$$= 0 \quad \text{if n is even.}$$

$b_n = 0$as f(x)is even function.

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{c} + a_2 \cos\frac{2\pi x}{c} + \cdots + b_1 \sin\frac{\pi x}{c} + b_2 \sin\frac{2\pi x}{c} + \cdots$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right] \quad \text{Ans.}$$

Example . Find Fourier half-range even expansion of the function,

$$f(x) = (-x/l) + 1, \quad 0 \le x \le l.$$

Solution. $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \left(-\frac{x}{l} + 1\right) dx$

$$= \frac{2}{l}\left[-\frac{x^2}{2l} + x\right]_0^l = \frac{2}{l}\left[-\frac{l^2}{2l} + l\right] = \frac{2l}{l}\left[-\frac{1}{2} + 1\right] = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(-\frac{x}{l} + 1\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(-\frac{x}{l} + 1\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \left(-\frac{1}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \frac{l}{n^2\pi^2}\cos n\,\pi + \frac{l}{n^2\pi^2}\right] = \frac{2}{l}\frac{l}{n^2\pi^2}[-(-1)^n + 1] = \frac{2}{n^2\pi^2}[1 - (-1)^n]$$

$$= \frac{4}{n^2\pi^2} \quad \text{when n is odd.}$$

$$= 0 \quad \text{when n is even.}$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} \ldots\right] \quad \text{Ans.}$$

Example . Find the Fourier half-range cosine series of the function

$$f(t) = 2t, \quad 0 < t < 1$$

$$= 2(2 - t), \quad 1 < t < 2$$

Solution. $f(t) = 2t, \quad 0 < t < 1$

$$= 2(2 - t), \quad 1 < t < 2$$

Example . Expand $f(x) = e^x$in a cosine series over $(0,1)$.

Solution. $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = 2(e-1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x + \cos n\pi x)\right]_0^1$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi + \cos n\pi) - \frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}[(-1)^n e - 1]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \cdots \ldots\ldots\ldots$$

$$e^x = e - 1 + 2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + \cdots\right] \quad \text{Ans.}$$

Exercise:

1. Find the Fourier series to represent $f(x)$, where

$$f(x) = -a \quad -c < x < 0$$

$$= a \quad 0 < x < c$$

$$\text{Ans. } \frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \cdots\right]$$

2. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

$$\text{Ans. } 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

3. Find the Fourier series of the function

$$f(x) = \begin{cases} -2 & \text{for } -4 < x < -2 \\ x & \text{for } -2 < x < 2 \\ 2 & \text{for } 2 < x < 4 \end{cases}$$

$$\text{Ans. } \frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \left(\frac{4}{3\pi} - \frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4} - \frac{2}{2\pi}\sin\frac{4\pi x}{4} + \cdots$$

4. Find the Fourier series to represent

$$f(x) = x^2 - 2 \quad \text{from } -2 < x < 2.$$

Ans. $-\frac{2}{3}-\frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}\ldots\right]$

Laplace Transforms

Introduction: Laplace transforms help in solving differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

Definition: Let f(t) be a function defined for all positive values of t, then

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

provided the integral exists, is called the Laplace Transform of f(t). It is denoted as

$$L[f(t)]=F(s)=\int_0^\infty e^{-st}f(t)dt$$

Important Formulae:

1. $L(1) = 1 / s$
2. $L(t^n)=\frac{n!}{s^{n+1}}$, when n = 0, 1, 2, 3, ...
3. $L(e^{at}) = 1 / (s - a)$ $(s > a)$
4. $L(\cosh at)=\frac{s}{(s^2-a^2)}$ $(s^2>a^2)$
5. $L(\sinh at)=\frac{a}{(s^2-a^2)}$ $(s^2>a^2)$
6. $L(\sin at)=\frac{a}{(s^2+a^2)}$ $(s>0)$
7. $L(\cos at)=\frac{s}{(s^2+a^2)}$ $(s>0)$

Properties of Laplace Transforms:

(1) Linearity Property:

$$L[a f_1(t)+b f_2(t)]=a L[f_1(t)]+b L[f_2(t)]$$

(2) First Shifting Theorem:

$$\text{If } L f(t)=F(s), \text{then } L[e^{at}f(t)]=F(s-a).$$

Important results:

$$L(e^{at}t^n)=\frac{n!}{(s-a)^{n+1}}$$

2. $L(e^{at}\cosh bt) = ((s-a))/[(s-a)^2 - b^2]$

3. $L(e^{at}\sinh bt) = b/[(s-a)^2 - b^2]$.

4. $L(e^{at}\sin bt) = b/[(s-a)^2 + b^2]$.

5. $L(e^{at}\cos bt) = (s-a)/[(s-a)^2 + b^2]$.

Examples:

1. Find the Laplace transform of $\cos^2 t$.

$$L(\cos^2 t) = L[1/2(\cos 2t + 1)] = 1/2[s/(s^2+4) + 1/s].$$

2. Find the Laplace Transform of $t^{-\frac{1}{2}}$.

$$L\left(t^{-1/2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{(\pi/s)}.$$

1. $L[t + t^2 + t^3]$ Ans. $1/s^2 + 2/s^3 + 6/s^4$

2. $L[\sin t\cos t]$ Ans. $1/(s^2+4)$

3. $L[t^3 e - 2t]$ Ans. $6/(s+2)^4$

4. $L[\sin^3 2t]$ Ans. $48/[(s^2+4)(s^2+36)]$

5. $L[\sin 2t\cos 3t]$ Ans. $2(s^2-5)/[(s^2+1)(s^2+25)]$

Laplace Transform Of the Derivative Of F(t):

$$L[f'(t)] = s\,L[f(t)] - f(0) \text{ where } L[f(t)] = F(s)$$

Proof: Integrating by parts,

$$L[f'(t)] = \int_0^\infty e^{-st}\, f'(t)dt$$

$$= [\,e^{-st}f(t)]_0^\infty - \int_0^\infty (-s\,e^{-st})f(t)dt$$

$$= -f(0) + s\,L\,f(t).$$

Note: Roughly, Laplace transform of derivative of f(t) corresponds to multiplication of the Laplace transform of f(t) by s.

Laplace Transform Of Derivative of Order n:

$$L[f^n(t)] = s^n\,L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

Proof by successive application of derivative formula:

$$L[f''(t)] = s\,L[f'(t)] - f'(0) = s^2 L[f(t)] - s\,f(0) - f'(0)$$

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s\,f'(0) - f''(0)$$

Laplace Transform of Integral of F(t):

$$L\left[\int_0^t f(t)\ dt\right] = \frac{1}{s}F(s), \text{where } L[f(t)] = F(s)$$

Proof: Let $\varphi(t) = \int_0^t f(t)\ dt$ with $\varphi(0) = 0$, so $\varphi'(t) = f(t)$.

$$L[\varphi'(t)] = s\,L[\varphi(t)] = L[\varphi(t)] = \left(\frac{1}{s}\right)L[f(t)].$$

Laplace Transform of t. F(t) (Multiplication By t)

$$L[t^n f(t)] = (-1)^n \left(\frac{d^n}{ds^n}\right)[F(s)]$$

Proof: Differentiating $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ w.r.t. s yields

$$\frac{d}{ds}[F(s)] = \int_0^\infty -t\,e^{-st} f(t)\,dt = -L[t\,f(t)].$$

Extending to n-th order yields the result.

Examples

1. Find the Laplace transform of t sinh at.

$$L[t\sinh at] = -\frac{d}{ds}\left[\frac{a}{(s^2-a^2)}\right] = \frac{2as}{(s^2-a^2)^2}.$$

2. Find the Laplace transform of t^2 cos at.

$$L[t^2\cos at] = \frac{d^2}{ds^2}\frac{s}{(s^2+a^2)^2} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}.$$

Laplace Transform of $(1/t)f(t)$ (**division by** t)

$$L\left[\left(\frac{1}{t}\right)f(t)\right] = \int_s^\infty F(s)\ ds$$

Proof: Integrating F(s) w.r.t s from s to ∞ gives

$$\int_s^\infty F(s)\ ds = \int_0^\infty e^{-st}\,[(1/t)\,f(t)]\,dt = L[(1/t)\,f(t)].$$

Examples:

1. Find the Laplace transform of $\frac{(\sin 2t)}{t.}$.

$$L[(\sin 2t)/t] = \int_s^\infty [\,2 / (s^2 + 4)\,]\,ds = \cot^{-1}(s/2)\,.$$

2. Find the Laplace transform of $\frac{(1-\cos t)}{t^2}$.

$$L\left[\frac{1-\cos t}{t^2}\right] = \cot^{-1} s + \left(\frac{s}{2}\right)\log\left[\frac{s^2}{(s^2+1)}\right].$$

Exercise :

1. $L[t \sin at]$ Ans. $\frac{2as}{(s^2+a^2)^2}$.

2. $L\left[\left(\frac{1}{t}\right)(1-e^t)\right]$ Ans. $\log\left[\frac{s-1}{s}\right]$

3. $L\left[\left(\frac{1}{t}\right)(e^{-at}-e^{-bt})\right]$ Ans. $\log\left[\frac{s+b}{s+a}\right]$

Unit Step Function: The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$$

Example: Express the following function in terms of unit step functions and find its Laplace transform: $f(t) = 8$ for $t < 2, f(t) = 6$ for $t > 2$.

Solution:

$$f(t) = 8 + 0\,(t < 2), 8 - 2\,(t > 2) = 8 - 2\,u(t-2)$$

$$L[f(t)] = 8\,L(1) - 2\,L[u(t-2)] = \left(\frac{8}{s}\right) - 2\left(\frac{e^{(-2s)}}{s}\right).$$

Example: Express the following function in terms of unit step function:

$f(t) = \begin{cases} t-1 & \text{, for } 1 < t < 2 \\ 3-t & \text{for } 2 < t < 3, \end{cases}$ find its Laplace transform.

Solution: Solution:

$$f(t) = (t-1)[u(t-1) - u(t-2)] + (t-3)[u(t-2) - u(t-3)$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$L[f(t)] = \left(\frac{e^{-s}}{s^2}\right) - 2\left(\frac{e^{-2s}}{s^2}\right) + \left(\frac{e^{-3s}}{s^2}\right).$$

Formula: Laplace Transform of Unit Step Function:

$$L[u(t-a)] = \frac{e^{-as}}{s}.$$

Second Shifting Theorem: If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{(-as)}F(s)$.

Theorem:

$$L[f(t)u(t-a)] = e^{(-as)}L[f(t+a)]$$

Example : Find the Laplace Transform of $t^2u(t-3)$.

$$L[t^2u(t-3)] = e^{(-3s)}[(2/s^3) + (6/s^2) + (9/s)] \qquad \text{(Ans.)}$$

Example: Represent $f(t) = \sin 2t$ for $2\pi < t < 4\pi$ and $f(t) = 0$ otherwise, in terms of unit step function and then find its Laplace transform.

$$L[f(t)] = \left(e^{(-2\pi s)} - e^{(-4\pi s)}\right) \cdot [2/(s^2 + 4)] \qquad \text{(Ans.)}$$

Exercise:

1. $f(t) = t - 1 (1 < t < 2)$, 0 otherwise — Ans $\left(e^{(-s)} - e^{(-2s)}\right)/s^2 - \left(e^{(-2s)}/s\right)$
2. $e^t\, u(t-1)$ — Ans. $e^{(-(s-1))}/(s-1)$
3. $t^2\, u(t-2)$ — Ans. $\left(e^{(-2s)}/s^3\right)(4s^2 + 4s + 2)$
4. $\sin t\, u(t-4)$ — Ans. $\left(e^{(-4s)}/(s^2+1)\right)[\cos 4 + s\sin 4]$
5. $f(t) = K(t-2)[U(t-2) - U(t-3)]$ — Ans. $(K/s^2)\left[e^{(-2s)} - (s+1)e^{(-3s)}\right]$

Impulse Function & Unit Impulse Function:

1. Unit Impulse Function: $\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$, with $\int_0^\infty \delta(t-a)\, dt = 1.$

2. Laplace Transform of Unit Impulse Function:

$L[\delta(t-a)] = e^{(-as)} L[f(t)\delta(t-a)]$

$= e^{(-as)} f(a) L[f(t)\delta'(t-a)]$

$= -d/dt\left[e^{(-st)} f(t)\right]|_{t=a}$

Exercise : Find the laplace transform :

1. $\int_0^\infty e^{-3t}\, \delta(t-4) dt$ — Ans. e^{-12}.
2. $\int_0^\infty \sin 2t\, \delta(t - \pi/4) dt$ — Ans. 1
3. $\int_{-\infty}^\infty e^{-3t}\, \delta'(t-2) dt$ — Ans. $3e^{(-6)}$
4. $\delta(t-4)/t$ — Ans. $e^{(-4s)}/4$
5. $e^{(-4t)}\delta(t-3)$ — Ans. $e^{(-3(s+4))}$

Periodic Functions: Let $f(t)$ be a periodic function with Period T, then:

$L[f(t)] = \frac{1}{(1 - e^{(-sT)})} \int_0^T e^{-st} f(t) dt.$

Example:

$f(t) = \frac{2t}{3}$ for $0 \leq t \leq 3.$

$$L[f(t)] = \frac{2[1-(1+3s)e^{-3s}]e^{-3s}}{3s^2(1-e^{-3s})} \qquad \text{(Ans.)}$$

Example (Half Wave Rectifier):

$$f(t) = \begin{cases} \sin\omega t & \text{for } 0 < t < \pi/\omega \\ 0 & \text{for } \pi/\omega < t < 2\pi/\omega \end{cases}$$

$$L[f(t)] = \omega/\left[(s^2+\omega^2)\left(1-e^{(-\pi s/\omega)}\right)\right] \qquad \text{(Ans.)}$$

Example (Saw Tooth Wave):

$f(t) = (k/T)t \quad$ for $\quad 0 < t < T.$

$$L[f(t)] = \frac{-ke^{(-sT)}}{s(1-e^{(-sT)})} + \frac{k}{Ts^2} \qquad \text{(Ans.)}$$

Example (Rectangular Wave)

$$f(t) = \begin{cases} A & \text{for } 0 < t < T/2 \\ -A & \text{for } T/2 < t < T \end{cases}$$

$$L[f(t)] = \frac{A}{s}\tanh\frac{sT}{4} \qquad \text{(Ans.)}$$

Example (Periodic Square Wave)

$f(t) = e^t \quad$ for $\quad 0 < t < 2\pi.$

$$L[f(t)] = \frac{1-e^{2\pi(1-s)}}{(1-e^{(-2\pi s)})(s-1)} \qquad \text{(Ans.)}$$

Exercise :

1. $f(t) = e^t$ for $0 < t < 2\pi.$ Ans. $\left(e^{2(1-s)\pi}-1\right)/[(1-s)(1-e^{-2\pi s})]$

2. Full wave rectified sin wave: $f(t) = \sin\omega t \qquad (0 < t < \pi/\omega)$

Ans. $\left(\omega/(s^2+\omega^2)\right)\coth(\pi s/2\omega)$

3. Staircase function: $f(t) = kn,\ np < t < (n+1)p$ Ans. $ke^{(ps)}/[s(1-e^{-ps})]$

4. $f(t) = t^2,\ 0 < t < 2, \quad f(t+2) = f(t)$

Ans. $(2-e^{-2s}-4se^{-2s}-4s^2e^{-2s})/[s^3(1-e^{-2s})]$

5. $f(t) = \begin{cases} t & ,0 < t < c \\ 2c-t, & c < t < 2c \end{cases}$ Ans. $(1/s^2)\tanh(cs/2)$

6. $f(t) = \begin{cases} \cos\omega t, & 0 < t < \pi/\omega \\ 0 \quad , & \pi/\omega < t < 2\pi/\omega \end{cases}$

Ans. $\left(s/(s^2+\omega^2)\right)\left(1-e^{-\pi s/\omega}\right)$

Convolution Theorem:

$$L\left[\int_0^t f_1(x)\, f_2(t-x)dx\right] = F_1(s)\cdot F_2(s)$$

Laplace Transform Of Bessel Functions $J_0(X)$ And $J_1(X)$:

$$L[J_0(t)] = 1/\sqrt{(s^2+1)}$$
$$L[J_0(at)] = 1/\sqrt{(s^2+a^2)}$$
$$L[J_1(x)] = 1 - s/\sqrt{(s^2+1)}$$

Evaluation Of Integrals:

Example: Evaluate $\int_0^t t\ e^{-3t} \sin t\, dt$.

(Ans.) $3/50$

Example : Evaluate $\int_0^\infty e^{-t}\ \frac{\sin t}{t} dt$ and $\int_0^\infty \frac{\sin t}{t} dt$.

$$\int_0^\infty e^{-t}\ \frac{\sin t}{t} dt = \pi/4 \qquad \text{(Ans.)}$$

$$\int_0^\infty \frac{\sin t}{t} dt = \pi/2 \qquad \text{(Ans.)}$$

Exercise

Evaluate the following by using Laplace Transform:

1. $\int_0^\infty te^{-4t} \sin t dt$ Ans. $8/289$
2. $\int_0^\infty e^{-2t}\ \frac{\sinh t \sin t}{t} dt$ Ans. $(1/2)\tan^{-1}(1/2)$
3. $\int_0^\infty \frac{\sin^2 t}{t^2} dt$ Ans. $\pi/2$
4. $\int_0^\infty \frac{e^{-t}-e^{-4t}}{t}$ Ans. $\log 4$

Formulae Of Laplace Transform:

S.No.	f(t)	F(s)
1.	$e^{(at)}$	$1/(s-a)$
2.	t^n	$\Gamma(n+1)\,/\,s^{(n+1)} or\ \ n!\,/\,s^{(n+1)}$
3.	$sinat$	$a/(s^2+a^2)$
4.	$cosat$	$s/(s^2+a^2)$
5.	$sinhat$	$a/(s^2-a^2)$
6.	$coshat$	$s/(s^2-a^2)$

7.	$U(t-a)$	$e^{(-as)}/s$
8.	$\delta(t-a)$	$e^{(-as)}$
9.	$e^{(bt)}sinat$	$a/[(s-b)^2+a^2]$
10.	$e^{(bt)}cosat$	$(s-b)/[(s-b)^2+a^2]$
11.	$(1/2a)sinat$	$s/(s^2+a^2)^2$
12.	$tcosat$	$(s^2-a^2)/(s^2+a^2)^2$
13.	$(1/2a^3)(sinat-atcosat)$	$1/(s^2+a^2)^2$
14.	$(1/2a)(sinat+atcosat)$	$s^2/(s^2+a^2)^2$

PROPERTIES OF LAPLACE TRANSFORM

S.No.	Property	f(t)	F(s)
1.	Scaling	$f(at)$	$(1/a)F(s/a), a>0$
2.	Derivative	$df(t)/dt$	$sF(s)-f(0), s>0$
		$d^2f(t)/dt^2$	$s^2F(s)-sf(0)-f'(0), s>0$
		$d^3f(t)/dt^3$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0), s>0$
3.	Integral	$\int_0^t f(t)dt$	$(1/s)F(s), s>0$
4.	Initial Value	$lim_{t\to 0}f(t)$	$lim_{s\to\infty}sF(s)$
5.	Final Value	$lim_{t\to\infty}f(t)$	$lim_{s\to 0}sF(s)$
6.	First shifting	$e^{(-at)}f(t)$	$F(s+a)$
7.	Second shifting	$f(t)U(t-a)$	$e^{(-as)}Lf(t+a)$
8.	Multiplication by t	$tf(t)$	$-d/dsF(s)$
		$t^nf(t)$	$(-1)^n(d^n/ds^n)F(s)$
9.	Division by t	$(1/t)f(t)$	$\int_s^\infty F(s)ds$
10.	Periodic function	$f(t)$	$\left[\int_0^T e^{(-st)}f(t)dt\right]/\left(1-e^{(-sT)}\right), f(t+T)=f(t)$
11.	Convolution	$f(t)*g(t)$	$F(s)G(s)$

Inverse Laplace Transforms: Now we obtain f(t) when F(s) is given, then we say that inverse Laplace transform of F(s) is f(t).

$$\text{If } Lf(t) = F(s), \text{then } L^{-1}F(s) = f(t).$$

where L^{-1} is called the inverse Laplace transform operator. From the application point of view, the inverse Laplace transform is very useful.

Important Formulae:

(1) $L^{-1}(1/s) = 1$.

(2) $L^{-1}(1/s^n) = t^{(n-1)}/(n-1)!$.

(3) $L^{-1}\big(1/(s-a)\big) = e^{(at)}$.

(4) $L^{-1}\big(s/(s^2-a^2)\big) = \cosh at$.

(5) $L^{-1}\big(1/(s^2-a^2)\big) = (1/a)\sinh at$.

(6) $L^{-1}\big(1/(s^2+a^2)\big) = (1/a)\sin at$.

(7) $L^{-1}\big(s/(s^2+a^2)\big) = \cos at$.

(8) $L^{-1}[F(s-a)] = e^{(at)}f(t)$.

(9) $L^{-1}(1/[(s-a)^2+b^2]) = (1/b)e^{(at)}\sin bt$.

(10) $L^{-1}\big((s-a)/[(s-a)^2+b^2]\big) = e^{(at)}\cos bt$.

(11) $L^{-1}(1/[(s-a)^2-b^2]) = (1/b)e^{(at)}\sinh bt$.

(12) $L^{-1}\big((s-a)/[(s-a)^2-b^2]\big) = e^{(at)}\cosh bt$.

(13) $L^{-1}(1/(s^2+a^2)^2) = (1/2a^3)(\sin at - at\cos at)$.

(14) $L^{-1}(s/(s^2+a^2)^2) = (1/2a)t\sin at$.

(15) $L^{-1}\big((s^2-a^2)/(s^2+a^2)^2\big) = t\cos at$.

(16) $L^{-1}(1) = \delta(t)$.

(17) $L^{-1}(s^2/(s^2+a^2)^2) = (1/2a)[\sin at + at\cos at]$.

Example : Find the inverse Laplace Transform of the following:

(i) $1/(s-2)$ Ans. $e^{(2t)}$

(ii) $1/(s^2-9)$ Ans. $(1/3)\sinh 3t$

(iii) $s/(s^2-16)$ Ans. $\cosh 4t$

(iv) $1/(s^2+25)$ Ans. $(1/5)\sin 5t$

(v) $s/(s^2+9)$ Ans. $\cos 3t$

Example. Find the inverse Laplace transform of:

1. $(3s-8)/(4s^2+25)$ Ans. $(3/4)\cos(5t/2) - (4/5)\sin(5t/2)$

2. $3(s^2-2)^2/(2s^5)$ Ans. $(3/2) - 3t^2 + (1/2)t^4$

3. $(2s-5)/(4s^2+25) + (4s-18)/(9-s^2)$

Ans. $(1/2)[\cos(5t/2) - \sin(5t/2)] - 4\cosh 3t + 6\sinh 3t$

4. $(5s-10)/(9s^2-16)$ Ans. $(5/9)\cosh(4t/3) - (5/6)\sinh(4t/3)$

5. $(1/4s) + 16/(1-s^2)$ Ans. $(1/4) - 16\sinh t$

6. $1/[(s-2)^2+1]$ Ans. $e^{(2t)}\sin t$

7. $(s-1)/[(s-1)^2+4]$ Ans. $e^t\cos 2t$

8. $1/[(s+3)^2-4]$ Ans. $(1/2)e^{(-3t)}\sinh 2t$

9. $(s+2)/[(s+2)^2-25]$ Ans. $e^{(-2t)}\cosh 5t$

10. $1/(2s-7)$ Ans. $(1/2)e^{(7t/2)}$

Multiplication by S

$$L^{-1}[sF(s)] = (d/dt)f(t) + f(0)\delta(t)$$

Example : Find the inverse Laplace transform of:

(i) $s/(s^2+1)$ Ans. $\cos t$

(ii) $s/(4s^2-25)$ Ans. $(1/4)\cosh(5t/2)$

(iii) $3s/(2s+9)$ Ans. $-(27/4)e^{(-11t/2)} + 3/2$

Exercise :

1. $s/(s+5)$ Ans. $-5e^{(-5t)} + 1$

2. $2s/(3s+6)$ Ans. $(2/3)\left[-2e^{(-2t)} + 1\right]$

3. $s/(2s^2-1)$ Ans. $(1/2)\cosh\left(t/\sqrt{2}\right)$

4. $s^2/(s^2+a^2)$ Ans. $-a\sin at + 1$

5. $(s^2+4)/(s^2+9)$ Ans. $-(5/3)\sin 3t + 1$

6. $[s^2/(s^2+4)^2]$ Ans. $(1/4)\sin 2t + (t/2)\cos 2t$

Division by s (multiplication by 1/s)

$L^{-1}[F(s)/s] = \int_0^t L^{-1}[F(s)]\,dt = \int_0^t f(t)\,dt.$

Example : Find the inverse Laplace transform of:

(i) $1/[s(s+a)]$ Ans. $(1/a)\left[1 - e^{(-at)}\right]$

(ii) $1/[s(s^2+1)]$ Ans. $1 - \cos t$

(iii) $(s^2+3)/[s(s^2+9)]$ Ans. $(1/3)[2\cos 3t + 1]$

Exercise : Find the inverse laplace transform:

1. $1/[2s(s-3)]$ Ans. $(1/2)\left[e^{(3t)}/3 - 1\right]$

2. $1/[s(s+2)]$ Ans. $\left(1 - e^{(-2t)}\right)/2$

3. $1/[s(s^2-16)]$ Ans. $(1/16)[\cosh 4t - 1]$

4. $1/[s(s^2+a^2)]$ Ans. $(1 - \cos at)/a^2$

5. $(s^2+2)/[s(s^2+4)]$ Ans. $\cos^2 t$

6. $1/[s^2(s+1)]$ Ans. $t-1+e^{(-t)}$

7. $1/[s^3(s^2+1)]$ Ans. $(t^2/2)+\cos t-1$

8. $1/\big(s(s^2+1)\big)$ Ans. $1-\cos t$

First Shifting Property:

$$\text{If } L^{-1}[F(s)]=f(t), \quad \text{then} \quad L^{-1}[F(s+a)]=e^{(-at)}L^{-1}[F(s)].$$

Example : Find the inverse Laplace transform of

(i) $1/(s+2)^5$ Ans. $e^{(-2t)}(t^4/24)$

(ii) $s/(s^2+4s+13)$ Ans. $e^{(-2t)}\cos 3t-(2/3)e^{(-2t)}\sin 3t$

(iii) $1/(9s^2+6s+1)$ Ans. $\big(te^{(-t/3)}\big)/9$

Exercise:

1. $(s+8)/(s^2+4s+5)$ Ans. $e^{(-2t)}(\cos t+6\sin t)$

2. $s/[(s+3)^2+4]$ Ans. $e^{(-3t)}(\cos 2t-1.5\sin 2t)$

3. $s/(s+7)^4$ Ans. $e^{(-7t)}(t^2/6)(3-7t)$

4. $(s+2)/(s^2-2s-8)$ Ans. $e^{t}(\cosh 3t+\sinh 3t)$

5. $s/(s^2+6s+25)$ Ans. $e^{(-3t)}[\cos 4t-(3/4)\sin 4t]$

6. $1/[2(s-1)^2+32]$ Ans. $(e^t/8)\sin 4t$

7. $(s-4)/[4(s-3)^2+16]$ Ans. $(1/4)e^{(3t)}\cos 2t-(1/8)e^{(3t)}\sin 2t$

Second Shifting Property:

$$L^{-1}\big[e^{(-as)}F(s)\big]=f(t-a)U(t-a)$$

Example : Find inverse Laplace transform of

(i) $e^{(-\pi s)}/(s+3)$ Ans. $e^{(-3(t-\pi))}U(t-\pi)$

(ii)$e^{(-s)}/(s+1)^3$ Ans. $e^{(-(t-1))}[(t-1)^2/2]U(t-1)$

Exercise :

1. $\big[se^{(-s/2)}+\pi e^{(-s)}\big]/(s^2+\pi^2)$ Ans. $[U(t-1/2)-U(t-1)]\sin\pi t$

2. $e^{(-s)}/(s+2)^3$ Ans. $e^{(-(t-2))}[(t-2)^2/2]U(t-2)$

3. $e^{(-2s)}/[(s+1)(s^2+2s+2)]$ Ans. $e^{(-(t-2))}[1-\cos(t-2)]U(t-2)$

4. $e^{(-s)}/\sqrt{(s+1)}$ Ans. $e^{(-(t-1))}/\sqrt{\big(\pi(t-1)\big)}U(t-1)$

5. $\big[e^{(-\pi s/2)}+e^{(-3\pi s/2)}\big]/(s^2+1)$ Ans. $\cot[U(t-3\pi/2)-U(t-\pi/2)]$

6. $e^{(-4s)}(s+2)/(s^2+4s+5)$ Ans. $e^{(-2(t-4))}\cos(t-4)U(t-4)$

7. $e^{(-as)}/s^2$ Ans. $f(t)=t-a$ when $t>a$; 0 when $t<a$

8. $e^{(-\pi s)}/(s^2+1)$ Ans. $-\text{sintu}(t-\pi)$

Inverse Laplace Transforms of Derivatives:

$$L^{-1}[(d/ds)F(s)] = -tL^{-1}[F(s)]$$
$$= -tf(t)$$
$$or \quad L^{-1}[F(s)] = -(1/t)L^{-1}[(d/ds)F(s)]$$

Exercise :

1. $\log(1+\omega^2/s^2)$ Ans. $-(2/t)\cos\omega t + 2$
2. $\log(1+1/s^2)$ Ans. $(2/t)[1-\cos t]$
3. $s/(1+s^2+s^4)$ Ans. $\left(2/\sqrt{3}\right)\sin\left(\sqrt{3}t/2\right)\sinh(t/2)$
4. $(s+1)/(s^2+6s+13)^2$ Ans. $\left(e^{(-3t)}/8\right)[2t\sin 2t + 2t\cos 2t - \sin 2t]$
5. $s/(s^2+a^2)^2$ Ans. $(t\sin at)/2a$
6. $s\log\left[s/\sqrt{(s^2+1)}\right] + \cot^{-1}s$ Ans. $(1-\cos t)/t^2$
7. $(1/2)\log[(s^2+b^2)/(s-a)^2]$ Ans. $\left(e^{(at)} - \cos bt\right)/t$
8. $\tan^{-1}(s+1)$ Ans. $-(1/t)e^{(-t)}\sin t$

Inverse Laplace Transform of Integrals:

$$L^{-1}\left[\int_s^\infty F(s)ds\right] = f(t)/t = (1/t)L^{-1}[F(s)]$$

Example: Find the inverse Laplace transforms of $\frac{(s+4)}{s(s-1)(s^2+4)}$.

Solution :Partial fraction decomposition

$$\frac{(s+4)}{s(s-1)(s^2+4)} = -\frac{1}{s} + \frac{1}{(s-1)} - \frac{1}{(s^2+4)}$$
$$\therefore L^{-1}\left[\frac{(s+4)}{s(s-1)(s^2+4)}\right] = L^{-1}\left[-\frac{1}{s} + \frac{1}{(s-1)} - \frac{1}{(s^2+4)}\right]$$
$$= -L^{-1}[1/s] + L^{-1}[1/(s-1)] - (1/2)L^{-1}[2/(s^2+2^2)]$$
$$= -1 + e^t - (1/2)\sin 2t \qquad \text{Ans.}$$

Example : Find the Laplace inverse of $s^2/[(s^2+a^2)(s^2+b^2)]$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}[s^2/(s^2+a^2)(s^2+b^2)]$$
$$= L^{-1}[a^2/(a^2-b^2)\cdot 1/(s^2+a^2) - b^2/(a^2-b^2)\cdot 1/(s^2+b^2)]$$
$$= 1/(a^2-b^2)L^{-1}[a^2/(s^2+a^2) - b^2/(s^2+b^2)]$$
$$= 1/(a^2-b^2)\left[a^2\left((1/a)sinat\right) - b^2\left((1/b)\sin bt\right)\right]$$
$$= 1/(a^2-b^2)[a\sin at - b\sin bt] \qquad \text{Ans.}$$

Exercise : Find the inverse transforms of

1. $(s^2+2s+6)/s^3$ Ans. $1+2t+3t^2$

2. $1/(s^2-7s+12)$ Ans. $e^{4t}-e^{3t}$

3. $(s+2)/(s^2-4s+13)$ Ans. $e^{2t}\cos 3t+(4/3)e^{2t}\sin 3t$

4. $(3s+1)/[(s-1)(s^2+1)]$ Ans. $e^t-2\cos t+\sin t$

5. $(11s^2-2s+5)/(2s^3-3s^2-3s+2)$ Ans. $2e^{-t}+5e^{2t}-(3/2)e^t/^2$

6. $(2s^2-6s+5)/[(s-1)(s-2)(s-3)]$ Ans. $(1/2)e^t-e^{2t}+(5/2)e^{3t}$

7. $(s-4)/[(s-4)^2+9]$ Ans. $e^{4t}\cos 3t$

8. $16/(s^2+2s+5)^2$ Ans. $e^{-t}(\sin 2t-2t\cos 2t)$

9. $1/[(s+1)(s^2+2s+2)]$ Ans. $e^{-t}(1-\cos t)$

10. $1/[(s-2)(s^2+1)]$ Ans. $(1/5)e^{2t}-(1/5)\cos t-(2/5)\sin t$

Inverse Laplace Transform by Convolution:

$$L\int_0^t f_1(x)*f_2(t-x)dx = F_1(s)\cdot F_2(s)$$

$$\int_0^t f_1(x)\cdot f_2(t-x)dx = L^{-1}[F_1(s)\cdot F_2(s)]$$

Example : Using the convolution theorem, find $L^{-1}[s^2/(s^2+a^2)(s^2+b^2)], a\neq b$.

Solution. We have

$$L(\cos at) = s/(s^2+a^2) \quad \text{and} \quad L(\cos bt) = s/(s^2+b^2)$$

Hence by the convolution theorem

$$L\int_0^t \cos ax\ \cos b(t-x)dx = s^2/[(s^2+a^2)(s^2+b^2)]$$

$$\text{Therefore, } L^{-1}[s^2/(s^2+a^2)(s^2+b^2)] = \int_0^t \cos ax\ \cos b(t-x)dx$$

$$= (1/2)\int_0^t [\cos(ax+bt-bx)+\cos(ax-bt+bx)]\,dx$$

$$= (1/2)\int_0^t \cos[(a-b)x+bt]dx + (1/2)\int_0^t \cos[(a+b)x+bt]dx$$

$$= [\sin[(a-b)x+bt]/2(a-b)]_0^t + [\sin[(a+b)x-bt]/2(a+b)]_0^t$$

$$= (\sin at-\sin bt)/[2(a-b)] + (\sin at+\sin bt)/[2(a+b)]$$

$$= (a\sin at-b\sin bt)/(a^2-b^2) \qquad \text{Ans.}$$

Example: Obtain $L^{-1}[1/s(s^2+a^2)]$.

Solution. $L^{-1}[1/s] = 1$ and $L^{-1}[1/(s^2+a^2)] = (\sin at)/a$

Hence by the convolution theorem

$$L\int_0^t 1 \cdot \frac{\sin a(t-x)}{a}\,dx = \frac{1}{s(s^2+a^2)}$$

$$L^{-1}[1/s(s^2+a^2)] = \int_0^t 1 \cdot \sin a(t-x)/a\,dx$$

$$= [-\cos(at-ax)/(-a^2)]_0^t$$

$$= (1/a^2)[1-\cos at] \qquad \text{Ans.}$$

Exercise: Find the inverse Laplace transform by convolution:

1. $s^2/(s^2+a^2)^2$ Ans. $(1/2)t\cos at + (1/2a)\sin at$

2. $1/(s^2+1)^3$ Ans. $(1/8)[(3-t^2)\sin t - 3t\cos t]$

Solution of Differential Equations by Laplace Transforms: Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants. The method will be clear from the following examples.

Example: Using Laplace transforms, find the solution of the initial value problem

$$y'' - 4y' + 4y = 64\sin 2t, \quad y(0) = 0, y'(0) = 1$$

Solution: $y'' - 4y' + 4y = 64\sin 2t \ldots (1)$

Taking Laplace transform of both sides of (1), we have

$$[s^2\bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = (64 \times 2)/(s^2+4) \ldots (2)$$

On putting the value of $y(0)$ and $y'(0)$in (2), we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = 128/(s^2+4)$$

$$(s^2 - 4s + 4)\bar{y} = 1 + 128/(s^2+4)$$

$$(s-2)^2\bar{y} = 1 + 128/(s^2+4)$$

$$\bar{y} = 1/(s-2)^2 + 128/[(s-2)^2(s^2+4)]$$

$$= 1/(s-2)^2 - 8/(s-2) + 16/(s-2)^2 + 8s/(s^2+4)$$

$$y = L^{-1}[-8/(s-2) + 17/(s-2)^2 + 8s/(s^2+4)]$$

$$y = -8e^{2t} + 17te^{2t} + 8\cos 2t \qquad \text{Ans.}$$

Example: Using the Laplace transforms, find the solution of the initial value problem:

$$y'' + 25y = 10\cos 5t, y(0) = 2, y'(0) = 0$$

Solution. Taking Laplace transforms of the given differential eq., we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 25\bar{y} = 10s/(s^2 + 25)$$

$$s^2\bar{y} - 2s + 25\bar{y} = 10s/(s^2 + 25)$$

$$(s^2 + 25)\bar{y} = 2s + 10s/(s^2 + 25)$$

$$\bar{y} = 2s/(s^2 + 25) + 10s/(s^2 + 25)^2$$

$$L^{-1}[\bar{y}] = L^{-1}[2s/(s^2 + 25) + 10s/(s^2 + 25)^2]$$

$$y = L^{-1}[2s/(s^2 + 25) + 10s/(s^2 + 25)^2]$$

$$= 2\cos 5t + L^{-1}[10s/(s^2 + 25)^2]$$

$$= 2\cos 5t + L^{-1}[(d/ds) - 5/(s^2 + 25)]$$

$$= 2\cos 5t + t\sin 5t \qquad \text{Ans.}$$

Example : Applying convolution, solve the following initial value problem:

$$y'' + y = \sin 3t, y(0) = 0, y'(0) = 0$$

Solution. $y'' + y = \sin 3t$

Taking Laplace transformation of both the sides, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = 3/(s^2 + 9) \ldots (1)$$

On putting the values of y(0), y'(0) in (1) we get

$$s^2\bar{y} + \bar{y} = 3/(s^2 + 9)$$

$$(s^2 + 1)\bar{y} = 3/(s^2 + 9)$$

$$\bar{y} = 3/[(s^2 + 1)(s^2 + 9)]$$

$$= (3/8)[1/(s^2 + 1) - 1/(s^2 + 9)]$$

Taking the inversion transform we get

$$y = (3/8)L^{-1}[1/(s^2 + 1)] - (3/8)L^{-1}[1/(s^2 + 9)]$$

$$y = (3/8)\sin t - (3/8) \times (1/3)\sin 3t$$

$$= (3/8)\sin t - (1/8)\sin 3t \qquad \text{Ans.}$$

Example : Using Laplace transforms, find the solution of the initial value problem

$$y'' + 9y = 6\cos 3t, y(0) = 2, y'(0) = 0$$

Solution. $y'' + 9y = 6\cos 3t \ldots (1)$

$$y(0) = 2, y'(0) = 0$$

Taking Laplace transform of (1), we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 9\bar{y} = 6s/(s^2 + 9) \ldots (2)$$

Putting the value of $y(0)$ and $y'(0)$in (2), we have

$$s^2\bar{y} - 2s + 9\bar{y} = 6s/(s^2 + 9)$$

$$(s^2 + 9)\bar{y} = 2s + 6s/(s^2 + 9)$$

$$\text{or } \bar{y} = 2s/(s^2+9) + 6s/(s^2+9)^2$$

$$y = L^{-1}[2s/(s^2+9)] + L^{-1}[6s/(s^2+9)^2]$$

$$= 2\cos 3t + 3L^{-1}[(d/ds) - 3/(s^2+9)]$$

$$= 2\cos 3t - L^{-1}[(d/ds)3/(s^2+9)]$$

$$= 2\cos 3t + t\sin 3t \qquad \text{Ans.}$$

Example : Solve the initial value problem $2y'' + 5y' + 2y = e^{-2t}, y(0) = 1, y'(0) = 1$, using the Laplace transforms.

Solution. $2y'' + 5y' + 2y = e^{-2t}, y(0) = 1, y'(0) = 1$

Taking the Laplace Transform of both sides, we get

$$2[s^2\bar{y} - sy(0) - y'(0)] + 5[s\bar{y} - y(0)] + 2\bar{y} = 1/(s+2) \ldots (1)$$

On substituting the values $y(0)$ and $y'(0)$in (1), we get

$$2[s^2\bar{y} - s - 1] + 5[s\bar{y} - 1] + 2\bar{y} = 1/(s+2)$$

$$[2s^2 + 5s + 2]\bar{y} - 2s - 2 - 5 = 1/(s+2)$$

$$\bar{y} = 1/[(s+2)(2s^2+5s+2)] + (2s+7)/(2s^2+5s+2)$$

$$= (1 + 2s^2 + 7s + 4s + 14)/[(2s^2+5s+2)(s+2)]$$

$$= (2s^2 + 11s + 15)/[(2s+1)(s+2)^2]$$

$$= (4/9)/(2s+1) - (11/9)/(s+2) - (1/3)/(s+2)^2$$

$$= (4/9)\cdot(1/2)/(s+1/2) - (11/9)/(s+2) - (1/3)/(s+2)^2$$

$$y = (2/9)e^{-t} - (11/9)e^{-2t} - (1/3)te^{-2t} \qquad \text{Ans.}$$

Example: Solve $d^2y/dx^2 + 2dy/dx + 5y = e^{-x}\sin x$ where $y(0) = 0, y'(0) = 1$.

Solution. $d^2y/dx^2 + 2dy/dx + 5y = e^{-x}\sin x$

Taking the Laplace Transforms of both the sides, we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = L(e^{-x}\sin x)$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = 1/[(s+1)^2 + 1] \ldots (1)$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$(s^2\bar{y} - 1) + 2(s\bar{y}) + 5\bar{y} = 1/(s^2+2s+2)$$

$$(s^2 + 2s + 5)\bar{y} = 1 + 1/(s^2+2s+2)$$

$$= (s^2 + 2s + 3)/(s^2+2s+2)$$

$$\bar{y} = (s^2 + 2s + 3)/[(s^2+2s+5)(s^2+2s+2)]$$

On resolving the R.H.S. into partial fractions, we get

$$\bar{y} = (2/3)/(s^2+2s+5) + (1/3)/(s^2+2s+2)$$

On inversion, we obtain

$$y = (2/3)L^{-1}[1/(s^2+2s+5)] + (1/3)L^{-1}[1/(s^2+2s+2)]$$

$$y = (1/3)L^{-1}[2/(s+1)^2 + (2)^2] + (1/3)L^{-1}[1/(s+1)^2 + (1)^2]$$

or $y = (1/3)e^{-x}\sin 2x + (1/3)e^{-x}\sin x$ or $y = (1/3)e^{-x}(\sin x + \sin 2x)$ Ans.

Example . Using Laplace transforms, find the solution of the initial value problem $y'' + 9y = 9\,u(t-3), y(0) = y'(0) = 0$ where $u(t-3)$ is the unit step functions.

Solution. $y'' + 9y = 9\,u(t-3) \ldots (1)$

Taking Laplace transform of (1), we have

$$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 9e^{-3s}/s \ldots (2)$$

Putting the value of $y(0) = 0$ and $y'(0) = 0$ in (2), we get

$$s^2\bar{y} + 9\bar{y} = 9e^{-3s}/s$$

$$(s^2 + 9)\bar{y} = 9e^{-3s}/s$$

$$\bar{y} = 9e^{-3s}/[s(s^2 + 9)]$$

$$y = L^{-1}[9e^{-3s}/s(s^2 + 9)]$$

$$L^{-1}[3/(s^2 + 9)] = \sin 3t$$

$$3L^{-1}[3/s(s^2 + 9)] = 3\int_0^t \sin 3t\,dt$$

$$= -[\cos 3t]_0^t = 1 - \cos 3t$$

$$y = L^{-1}[9e^{-3s}/s(s^2 + 9)]$$

$$y = [1 - \cos 3(t-3)]u(t-3)$$ Ans.

EXERCISE

Solving the following differential equations:

1. $d^2y/dx^2 + y = 0, \text{where } y = 1 \text{ and } dy/dx = -1 \text{ at } x = 0.$

$$\text{Ans. } y = \cos x - \sin x$$

2. $d^2y/dx^2 - 4y = 0, \text{where } y = 0 \text{ and } dy/dx = -6 \text{ at } x = 0.$

$$\text{Ans. } y = -(3/2)e^{2x} + (3/2)e^{-2x}$$

3. $d^2y/dx^2 + y = 0, \text{where } y = 1, dy/dx = 1 \text{ at } x = 0.$

$$\text{Ans. } y = \sin x + \cos x$$

4. $d^2y/dx^2 + 2dy/dx + 5y = 0, \text{where } y = 2, dy/dx = -4 \text{ at } x = 0.$

$$\text{Ans. } y = e^{-x}(2\cos 2x - \sin 2x)$$

5. $d^3y/dx^3 + 2d^2y/dx^2 - dy/dx - 2y = 0, \text{given } y = 0,\ dy/dx = 0, d^2y/dx^2 = 6 \text{ at } x = 0.$ Ans. $y = e^x - 3e^{-x} + 2e^{-2x}$

6. $d^2y/dx^2 + y = 3\cos 2x, \text{where } y = dy/dx = 0 \text{ at } x = 0.$

$$\text{Ans. } y = \cos x - \cos 2x$$

7. $d^2y/dx^2 + dy/dx - 2y = 1 - 2x, \text{given } y = 0, dy/dx = 4 \text{ at } x = 0.$

$$\text{Ans. } y = e^{x} - e^{-2x} + x$$

8. $d^2y/dx^2 - 3dy/dx + 2y = 4e^{2x}$, given $y = -3$, and $dy/dx = 5$ at $x = 0$.

$$\text{Ans. } y = -7e^{x} + 4e^{2x} + 4xe^{2x}$$

9. $d^2y/dx^2 - 3dy/dx + 2y = 4x + e^{2x}$, where $y = 1, dy/dx = -1$ at $x = 0$.

$$\text{Ans. } y = 3 + 2x + (1/2)e^{3x} - 2e^{2x} - (1/2)e^{x}$$

10. $d^3y/dx^3 + 2d^2y/dx^2 - dy/dx - 2y = 0$, where $y = 1, dy/dx = 2$, $d^2y/dx^2 = 2$ at $x = 0$. $\text{Ans. } (5/3)e^{x} - e^{-x} + (1/3)e^{-2x}$

11. $(D^2 - D - 2)x = 20\sin 2t$, $x_0 = -1$, $x_1 = 2$.

$$\text{Ans. } x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$$

12. $(D^3 + D^2)x = 6t^2 + 4, x(0) = 0, x'(0) = 2, x''(0) = 0$

$$\text{Ans. } x = (1/2)t^4 - 2t^3 + 8t^2 - 16t + 16 - 16e^{-t}$$

13. $d^2x/dt^2 - 2dx/dt + x = e^{t}$, where $x(0) = 2$, $dx/dt = -1$ at $t = 0$.

$$\text{Ans. } x = 2e^{t} - 3te^{t} + (1/2)t^2e^{t}$$

14. $(D^2 + n^2)x = a\sin(nt + \alpha)$ where $x = Dx = 0$ at $t = 0$.

$$\text{Ans. } x = a_n \cos\alpha(\sin nt - nt\cos nt) + (a\sin 2\alpha/2n)(t\sin nt)$$

15. $y'' + 2y' + y = te^{-t}$ if $y(0) = 1, y'(0) = -2$. $\text{Ans. } y = (1 - t + t^3/6)e^{-t}$

16. $d^2y/dx^2 + y = x\cos 2x, where\ \ y = dy/dx = 0\ at x = 0$.

$$\text{Ans. } \quad y = (4/9)\sin 2x - (5/9)\sin x - (1/3)x\cos 2x$$

17. $d^3y/dx^3 - 3d^2y/dx^2 + 3dy/dx - y = x^2e^{2x}$, where $y = 1, dy/dx = 0, d^2y/dx^2 = -2$ at $x = 0$. $\text{Ans. } y = e^{2x}(x^2 - 6x + 12) - e^{x}(15x^2 + 7x + 11)$

18. $y'' + 4y' + 3y = t, t > 0$; given that $y(0) = 0$ and $y'(0) = 1$.

$$\text{Ans. } y = -4/9 + t/6 + e^{-t} - (5/9)e^{-3t}$$

19. $y'' + 2y = r(t), y(0) = 0, y'(0) = 0$ where $r(t) = 0$, $t \geq 1; 1, 0 \leq t < 1$.

$$\text{Ans. } y = 1/2 - (1/2)\cos\sqrt{2}t$$

20. $d^2y/dx^2 + y = u(t - \pi) - u(t - 2\pi), y(0) = y'(0) = 0$.

$$\text{Ans. } y = (1 + \cos t)u(t - \pi) - (1 - \cos t)u(t - 2\pi)$$

Solution Of Simultaneous Differential Equations By Laplace Transforms: Simultaneous differential equations can be also solved by Laplace Transform method.

Example : Solve $dx/dt + y = 0$ and $dy/dt - x = 0$under the condition $x(0) = 1, y(0) = 0$.

Solution. $x' + y = 0 \ldots (1)$

$$y' - x = 0 \ldots (2)$$

Taking the Laplace transform of (1) and (2) we get

$$[s\bar{x} - x(0)] + \bar{y} = 0 \ldots (3)$$

$$[s\bar{y} - y(0)] - \bar{x} = 0 \ldots (4)$$

On substituting the values of x(0) and y(0) in (3) and (4) we get

$$s\bar{x} - 1 + \bar{y} = 0 \ldots (5)$$

$$s\bar{y} - \bar{x} = 0 \ldots (6)$$

Solving (5) and (6) for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = s/(s^2 + 1), \bar{y} = 1/(s^2 + 1)$$

On inversion, we obtain

$$x = L^{-1}[s/(s^2 + 1)], y = L^{-1}[1/(s^2 + 1)]$$

$$x = \cos t, y = \sin t \qquad \text{Ans.}$$

Example: Using Laplace transforms, solve the differential equations

$$(D + 1)y_1 + (D - 1)y_2 = e^{-t}$$

$$(D + 2)y_1 + (D + 1)y_2 = e^{t}$$

where $D = d/dt$ and $y_1(0) = 1, y_2(0) = 0$.

Solution. $(D + 1)y_1 + (D - 1)y_2 = e^{-t} \ldots (1)$

$(D + 2)y_1 + (D + 1)y_2 = e^{t} \ldots (2)$

Multiply (1) by $(D + 1)$ and (2) by $(D - 1)$ we get

$$(D + 1)^2 y_1 + (D^2 - 1)y_2 = (D + 1)e^{-t} \ldots (3)$$

$$(D - 1)(D + 2)y_1 + (D^2 - 1)y_2 = (D - 1)e^{t} \ldots (4)$$

Subtracting (4) from (3) we get

$$(D^2 + 2D + 1 - D^2 - D + 2)y_1 = (-e^{-t} + e^{-t}) - (e^{t} - e^{t})$$

or $(D + 3)y_1 = 0$ or $Dy_1 + 3y_1 = 0$

Taking Laplace transforms we have

$$s\bar{y}_1 - y_1(0) + 3\bar{y}_1 = 0$$

$$(s + 3)\bar{y}_1 = 1 \quad \text{or} \quad \bar{y}_1 = 1/(s + 3)$$

$$y_1 = e^{-3t}$$

Laplace Transformation

$$y = -\tfrac{1}{2}L^{-1}[1/(s - 1)] + \tfrac{1}{2}L^{-1}[s/(s^2 + 1)] - \tfrac{1}{2}L^{-1}[1/(s^2 + 1)] + L^{-1}[s/(s^2 + 1)^2]y$$

$$= -\tfrac{1}{2}e^{t} + \tfrac{1}{2}\cos t - \tfrac{1}{2}\sin t + \tfrac{1}{2}t\sin t$$

$$y = ½[-e^t - \sin t + \cos t + t\sin t] \qquad \textbf{Ans.}$$

Example: Using the Laplace transforms, solve the initial value problem

$$y_1'' = y_1 + 3y_2$$

$$y_2'' = 4y_1 - 4e^t$$

$$y_1(0) = 2, y_1'(0) = 3, y_2(0) = 1, y_2'(0) = 2.$$

Solution:

$$y_1'' = y_1 + 3y_2 \ldots (1)$$

$$y_2'' = 4y_1 - 4e^t \ldots (2)$$

Taking the Laplace transform of (1) and (2):

$$s^2\bar{y}_1 - sy_1(0) - y_1'(0) = \bar{y}_1 + 3\bar{y}_2 \ldots (3)$$

$$s^2\bar{y}_2 - sy_2(0) - y_2'(0) = 4\bar{y}_1 - 4/(s-1) \ldots (4)$$

Putting the value of $y_1(0), y_1'(0), y_2(0), y_2'(0)$ in (3) and (4), we get:

$$s^2\bar{y}_1 - 2s - 3 = \bar{y}_1 + 3\bar{y}_2$$

$$\text{or} \quad (s^2 - 1)\bar{y}_1 - 3\bar{y}_2 = 2s + 3 \ldots (5)$$

$$s^2\bar{y}_2 - s - 2 = 4\bar{y}_1 - 4/(s-1)$$

$$\text{or} \quad 4\bar{y}_1 - s\bar{y}_2 = 4/(s-1) - s - 2 \ldots (6)$$

On solving (5) and (6), we get:

$$\bar{y}_1 = [(2s-3)(s^2+3)(s+2)]/[(s-1)(s^2+3)(s^2-4)]$$

$$= (2s-3)/[(s-1)(s-2)] = 1/(s-1) + 1/(s-2)$$

$$y_1 = e^t + e^{2t}$$

$$\bar{y}_2 = [(s+2)(s^2+3)]/[(s^2+3)(s^2-4)]$$

$$= 1/(s-2)$$

$$\Rightarrow y_2 = e^{2t} \qquad \text{Ans.}$$

Exercise : Solve the following:

1. $dx/dt + 4y = 0, dy/dt - 9x = 0.$ Given $x = 2$ and $y = 1$ at $t = 0.$

$$\text{Ans. } x = -\frac{2}{3}\sin 6t + 2\cos 6t, \qquad y = \cos 6t + 3\sin 6t$$

2. $4(dy/dt) + (dx/dt) + 3y = 0,\ 3(dx/dt) + 2x + (dy/dt) = 1$ under the condition $x = y = 0$ at $t = 0.$

$$\text{Ans. } x = \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-(6/11)t}\ , \qquad y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-(6/11)t}$$

3. $dx/dt + 5x - 2y = t, dy/dt + 2x + y = 0$ being given $x = y = 0$ when $t = 0$.

$$\text{Ans. } x = \frac{-1}{27}(1 + 6t)e^{-3t} + \frac{1}{27}(1 + 3t), \qquad y = \frac{-2}{27}(2 + 3t)e^{-3t} - \frac{2t}{9} + \frac{4}{27}$$

4. $dx/dt + y = \sin t, dy/dt + x = \cos t$

given that $x = 2$, and $y = 0$ when $t = 0$. Ans. $x = e^{t} + e^{-t}, y = e^{-t} - e^{t} + \sin t$

5. The small oscillations of a certain system with two degrees of freedom are given by the equations:

$$D^2x + 3x - 2y = 0, \qquad D^2x + D^2y - 3x + 5y = 0\ , \text{if } x = 0, y = 0, Dx = 3,$$

$$Dy = 2 \quad \text{when} \quad t = 0.$$

$$\text{Ans. } x = \frac{-11}{4}\sin t + \frac{1}{12}\sin 3t, \quad y = \frac{11}{4}\sin t - \frac{1}{4}\sin 3t.$$

6. $3(dx/dt) + 3(dy/dt) + 5x = 25\cos t,$

$$2(dx/dt) - 3(dy/dt) = 5\sin t \quad \text{with} \quad x(0) = 2, y(0) = 3.$$

$$\text{Ans. } x = 2\cos t + 3\sin t, \qquad y = 3\cos t + 2\sin 2t.$$

Fourier Integrals And Their Applications:

Introduction: Expansion of a function $f(x)$ of period $2l$ into a Fourier series is well known. The Fourier integral was first introduced by Fourier as an attempt to generalize his results from finite intervals to infinite intervals. In this chapter we propose to discuss Fourier integral representation and Fourier transfors, followed by applications involving the Fourier transforms.

Dirichlet's Conditions:

Let $f(x)$ satisfy the following conditions:

- (i) $f(x)$ is defined in the interval $-l < x < l$
- (ii) $f(x)$ and $f'(x)$ are sectionally continuous in $-l < x < l$
- (iii) $f(x)$ is periodic with period $2l$.

Then the above conditions are called Dirichlet's conditions.

Fourier integral theorem (or formula). Fourier integral representation

Let $F(x)$ satisfy the following conditions:

- (i) $f(x)$ satisfies the Dirichlet's conditions in every finite interval $-l \le x \le l$.

- (ii) $\int_{-\infty}^{\infty} |f(x)| dx$ converges, i.e., $f(x)$ is absolutely integrable in $-\infty < x < \infty$.

Then Fourier integral theorem states that

$$f(x) = \int_0^{\infty} \{A(s) \cos s\,x + B(s) \sin s\,x\}\, ds, \quad \dots (1)$$

where

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \cos s\,u\, du \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \sin s\,u\, du \quad \dots (2)$$

Fourier integral theorem can also be re-written as

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u) \cos s\,(x-u)\, du\, ds. \quad \dots (3)$$

The result (1) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, $f(x)$ is replaced by $\{f(x+0) + f(x-0)\}/2$. As for Fourier series, the above conditions are sufficient but not necessary.

The right side of (1) is sometimes called the Fourier integral expansion of $f(x)$, or briefly Fourier integral.

Different forms of Fourier integral theorem (formula):
(i) General form:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \left\{\int_0^{\infty} \cos s\,(x-u)\, ds\right\} du. \quad \dots (1)$$

Proof. When x is a point of continuity of $f(x)$, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} f(u) \cos s\,(x-u)\, du\right\} ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) \left\{\int_{-\infty}^{\infty} \cos s\,(x-u)\, ds\right\} du$$

[changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) \left\{2\int_0^{\infty} \cos s\,(x-u)\, ds\right\} du$$ using a property of definite integral, namely,

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^{a} f(x)\, dx, \text{ iff } f(-x) = f(x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \left\{\int_{s=0}^{\infty} \cos s\,(x-u)\, ds\right\} du.$$

(ii) Another general form: If $f(x)$ is continuous at x, then

$$f(x) = \int_0^\infty [A(s)\cos s\,x + B(s)\sin s\,x]\,ds, \quad \ldots(2)$$

where

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos s\,u\,du \quad \ldots(3)$$

and

$$B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin s\,u\,du. \quad \ldots(4)$$

Proof: We have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_0^\infty \cos s\,(x-u)\,ds\right\}du = \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(u)\cos s\,(x-u)\,du\right\}ds$$

[changing the order of integration]

$$= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(u)(\cos s\,x\cos s\,u + \sin s\,x\sin s\,u)\,du\right\}ds$$

$$= \int_0^\infty \left[\cos s\,x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos s\,u\,du\right\} + \sin s\,x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin s\,u\,du\right\}\right]ds$$

$$= \int_0^\infty [A(s)\cos s\,x + B(s)\sin s\,x]\;ds, \quad \text{using (3) and (4)}$$

(iii) Fourier sine integral formula: If $f(x)$ is an odd function, then

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin s\,x\left\{\int_0^\infty f(u)\sin s\,u\,du\right\}ds. \quad \ldots(5)$$

Proof. Proceed yourself as in form (ii) and show that

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos s\,x\left\{\int_{-\infty}^{\infty} f(u)\cos s\,u\,du\right\} + \sin s\,x\left\{\int_{-\infty}^{\infty} f(u)\sin s\,u\,du\right\}\right]ds \quad \ldots(6)$$

Since $f(x)$ is an odd function, it follows that $f(u)\cos s\,u$ is an odd function and $f(u)\sin s\,u$ is an even function and so by a property of definite integrals, we have

$$\int_{-\infty}^{\infty} f(u)\cos s\,u\,du = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(u)\sin s\,u\,du = 2\int_0^\infty f(u)\sin s\,u\,du.$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \sin s\,x\left\{\int_0^\infty f(u)\sin s\,u\,du\right\}ds.$$

(iv) Fourier cosine integral formula: If $f(x)$ is an even function, then

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos s\,x\left\{\int_0^\infty f(u)\cos s\,u\,du\right\}ds. \quad \ldots(7)$$

Proof: Proceed yourself as in part (ii) and show that

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos s\,x\left\{\int_{-\infty}^{\infty} f(u)\cos s\,u\,du\right\} + \sin s\,x\left\{\int_{-\infty}^{\infty} f(u)\sin s\,u\,du\right\}\right]ds \quad \ldots(8)$$

Since $f(x)$ is an even function, it follows that $f(u)\cos s\,u$ is an even function and $f(u)\sin s\,u$ is an odd function and hence we have

$$\int_{-\infty}^{\infty} f(u)\cos s\,u\,du = 2\int_0^{\infty} f(u)\cos s\,u\,du \quad \text{and} \quad \int_{-\infty}^{\infty} f(u)\sin s\,u\,du = 0.$$

$$\therefore \qquad \Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty}\cos s\,x\int_0^{\infty} f(u)\cos s\,u\,du.$$

(v) Complex or exponential form of Fourier integral theorem:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds. \quad \ldots(9)$$

Proof. From general form (i) of Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_0^{\infty}\cos s\,(x-u)\,ds\right\}du$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\cos s\,(x-u)\,du\right\}ds, \text{ changing the order of integration}$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\frac{e^{is(x-u)} + e^{-is(x-u)}}{2}\,du\right\}ds$$

$$\therefore f(x) = \frac{1}{2\pi}\int_0^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} f(u)e^{-isu}\,du\right\}ds + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds \quad \ldots(10)$$

Substituting $s = -y$ and $ds = -dy$ in the first integral only, (10) reduces to

$$f(x) = \frac{1}{2\pi}\int_0^{-\infty} e^{-iyx}\left\{\int_{-\infty}^{\infty} f(u)e^{iyu}\,du\right\}(-dy) + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-iyx}\left\{\int_{-\infty}^{\infty} f(u)e^{iyu}\,du\right\}dy + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}\,du\right\}ds.$$

Note: In (10), substituting $s = -y$ and $ds = -dy$ in the second integral in place of first integral and proceeding as before, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} f(u)e^{-isu}\,du\right\}ds, \quad \ldots(11)$$

which is another form of complex form of Fourier integral formula.

Solved Examples:

- (i) $\int_0^\infty e^{-ax}\cos b\,x\,dx = \frac{a}{a^2+b^2}$
- (ii) $\int_0^\infty e^{-ax}\sin b\,x\,dx = \frac{b}{a^2+b^2}$

Example: Using Fourier cosine integral formula, prove that $\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1}\,d\lambda = \frac{\pi}{2}e^{-x}, x \geq 0.$

Soution:. Fourier cosine integral formula is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \left\{\int_0^\infty f(u)\cos\lambda u\,du\right\}d\lambda \quad \ldots(1)$$

Take $f(x) = e^{-x}$ so that $f(u) = e^{-u}$. $\quad \ldots(2)$

Using (2), (1) reduces to

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \cos\lambda x \left\{\int_0^\infty e^{-u}\cos\lambda u\,du\right\}d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \cdot \frac{1}{1+\lambda^2}\,d\lambda, \quad \text{as } \int_0^\infty e^{-ax}\cos b\,x\,dx = \frac{a}{a^2+b^2}$$

$$\therefore \int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-x}$$

Example: Show that $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(u^2+2)\cos ux}{u^4+4}\,du, x > 0$

Solution: Fourier cosine integral formula is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos u\,x \left\{\int_0^\infty f(v)\cos u\,v\,dv\right\}du, \quad \ldots(1)$$

Taking $f(x) = e^{-x}\cos x$ so that $f(v) = e^{-v}\cos v$, (1) becomes

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \cos u\,x \left\{\int_0^\infty e^{-v}\cos v\cos u\,v\,dv\right\}du$$

$$= \frac{1}{\pi}\int_0^\infty \cos u\,x \left\{\int_0^\infty e^{-v}(2\cos u\,v\cos v)\,dv\right\}du$$

$$= \frac{1}{\pi}\int_0^\infty \cos u\,x \left[\int_0^\infty e^{-v}\{\cos(u+1)\,v + \cos(u-1)\,v\}\,dv\right]du$$

$$= \frac{1}{\pi}\int_0^\infty \cos u\,x \left[\int_0^\infty e^{-v}\cos(u+1)\,v\,dv + \int_0^\infty e^{-v}\cos(u-1)\,v\,dv\right]du$$

$$= \frac{1}{\pi}\int_0^\infty \cos u\,x \left[\frac{1}{1+(u+1)^2} + \frac{1}{1+(u-1)^2}\right]du, \quad \text{as } \int_0^\infty e^{-ax}\cos b\,x\,dx = \frac{a}{a^2+b^2}$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{(u^2+2-2u)+(u^2+2+2u)}{(u^2+2+2u)(u^2+2-2u)} \cos u\,x\,dx = \frac{1}{\pi}\int_0^{\infty} \frac{2(u^2+2)\cos u\,x}{(u^2+2)^2-(2u)^2}\,dx$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{(u^2+2)\cos u\,x}{u^4+4}$$

Example: Using Fourier sine integral formula, show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^{\infty} \frac{u \sin u\,x\,du}{(u^2+a^2)(u^2+b^2)}, \quad a \geq 0, b \geq 0.$$

Solution: Fourier sine integral formula is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin u\,x \left\{\int_0^{\infty} f(v) \sin u\,v\,dv\right\} du \quad \ldots (1)$$

Taking $f(x) = e^{-ax} - e^{-bx}$ so that $f(v) = e^{-av} - e^{-bv}$, (1) reduces to

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^{\infty} \sin u\,x \left\{\int_0^{\infty} \left(e^{-av} - e^{-bv}\right) \sin u\,v\,dv\right\} du$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin u\,x \left\{\int_0^{\infty} e^{-av} \sin u\,v\,dv - \int_0^{\infty} e^{-bv} \sin u\,v\,dv\right\} du$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin u\,x \left[\frac{u}{a^2+u^2} - \frac{u}{b^2+u^2}\right] du = \frac{2}{\pi}\int_0^{\infty} \frac{u(b^2-a^2)\sin u\,x\,du}{(a^2+u^2)(b^2+u^2)}$$

$$\therefore e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^{\infty} \frac{u \sin u\,x}{(a^2+u^2)(b^2+u^2)}\,du.$$

Example: Suppose a function f(x) is given by

$$f(x) = \begin{cases} 0, & \text{when } x \leq 0 \text{ or } x \geq \pi \\ \sin x, & \text{when } 0 \leq x \leq \pi. \end{cases}$$

Using Fourier integral formula, show that

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos u\,(\pi - x) + \cos u\,x}{1-u^2}\,du$$

Solution: Fourier integral formula is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \left\{\int_{-\infty}^{\infty} f(v) \cos u\,(v-x)\,dv\right\} du, \quad -\infty < x < \infty. \quad \ldots (1)$$

By given value of f(x), we have

$$f(v) = \begin{cases} 0, & v \leq 0 \text{ or } v \geq \pi \\ \sin v, & 0 \leq v \leq \pi \end{cases} \quad \ldots (2)$$

$$\therefore \int_{-\infty}^{\infty} f(v) \cos u\,(v-x)\,dv$$

$$= \int_{-\infty}^{0} f(v)\cos u\,(v-x)\,dv + \int_{0}^{\pi} f(v)\cos u\,(v-x)\,dv + \int_{\pi}^{\infty} f(v)\cos u\,(v-x)\,dv$$

$$= \int_{0}^{\pi} \sin v \cos u\,(v-x)\,dv, \quad \text{using given values of } f(v) \text{ from (2)}$$

$$= \frac{1}{2}\int_{0}^{\pi} [\sin\{(u+1)v-ux\} - \sin\{(u-1)v-ux\}]\,dv, \quad \text{as } 2\cos A \sin B$$

$$= \sin(A+B) - \sin(A-B)$$

$$= \frac{1}{2}\left[\frac{-\cos\{(u+1)v-ux\}}{u+1} + \frac{\cos\{(u-1)v-ux\}}{u-1}\right]_{v=0}^{v=\pi} = \frac{1}{2}\left[\frac{\cos C}{u-1} - \frac{\cos D}{u+1}\right]_{v=0}^{\pi},$$

[Taking $C = (u-1)v - ux$ and $D = (u+1)v - ux$]

$$= \frac{1}{2}\left[\frac{(u+1)\cos C - (u-1)\cos D}{(u-1)(u+1)}\right]_{v=0}^{\pi} = \left[\frac{u(\cos C - \cos D) + (\cos C + \cos D)}{2(u^2-1)}\right]_{v=0}^{\pi}$$

$$= \left[\frac{2u\sin\{(C+D)/2\}\sin\{(D-C)/2\} + 2\cos\{(C+D)/2\}\cos\{(D-C)/2\}}{2(u^2-1)}\right]_{v=0}^{\pi}$$

$$= \{1/(u^2-1)\}[u\sin(v-x)\,u\sin v + \cos(v-x)\,u\cos v]_{v=0}^{\pi}, \quad \text{putting values of C and D}$$

$$= [\cos(\pi - x)\,u\cos\pi - \cos(-xu)]/(u^2-1) = [\cos u\,(\pi - x) + \cos u\,x]/(1-u^2)$$

Thus,

$$\int_{-\infty}^{\infty} f(v)\cos u\,(v-x)\,dv = \frac{\cos u\,(\pi - x) + \cos u\,x}{1-u^2} \quad \ldots(3)$$

Using (4), (1) reduces to

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos u\,(\pi - x) + \cos u\,x}{1-u^2}\,du.$$

Example : Express the function $f(x) = \begin{cases} 1, & \text{when } |x| \le 1 \\ 0, & \text{when } |x| > 1 \end{cases}$ as a Fourier integral.

Hence evaluate $\int_0^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$.

Solution: Fourier integral formula is given by

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty}\left\{\int_{u=-\infty}^{\infty} f(u)\cos s\,(u-x)\,du\right\}ds. \quad \ldots(1)$$

Re-writing the given function f(x), we have

$$f(u) = \begin{cases} 1, & \text{when } -1 \le u \le 1 \\ 0, & u < -1 \text{ or } u > 1. \end{cases} \quad \ldots(2)$$

Now,

$$\int_{-\infty}^{\infty} f(u)\cos s\,(u-x)\,du$$

$$= \int_{-\infty}^{-1} f(u)\cos s\,(u-x)\,du + \int_{-1}^{1} f(u)\cos s\,(u-x)\,du + \int_{1}^{\infty} f(u)\cos s\,(u-x)\,du$$

$$= 0 + \int_{-1}^{1} \cos s\,(u - x)\,du + 0, \text{ using (2)}$$

$$= \left[\frac{\sin s\,(u-x)}{s}\right]_{u=-1}^{1} = \frac{\sin s\,(1-x) - \sin s\,(-1-x)}{s} = \frac{\sin s\,(1-x) + \sin s\,(1+x)}{s}$$

$$= \frac{2 \sin s \cos s\,x}{s} \quad \ldots (3)$$

Then using (3), (1) reduces to

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty} \frac{2 \sin s \cos s\,x}{s}\,ds = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin \lambda \cos \lambda\,x}{\lambda}\,d\lambda. \quad \ldots (4)$$

This is the required representation of f(x) as Fourier integral.

To evaluate the given integral. Re-writing (4), we have

$$\int_{0}^{\infty} \frac{\sin \lambda \cos \lambda\,x}{\lambda}\,d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad \text{[Using the given value of f(x).]}$$

Example: Express $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate $\int_0^{\infty} \frac{1-\cos \pi\lambda}{\lambda} \sin x\,\lambda\,d\lambda$.

Solution: Re-writing the given function f(x), we have

$$f(u) = \begin{cases} 1 & \text{for } 0 \le u \le \pi \\ 0 & \text{for } u > \pi. \end{cases} \quad \ldots (1)$$

Using variable λ in place of s in form (iii) of Art. 1.4, Fourier sine integral formula is

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin \lambda\,x \left\{\int_{0}^{\infty} f(u) \sin \lambda\,u\,du\right\} d\lambda$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \sin \lambda\,x \left\{\int_{0}^{\pi} f(u) \sin \lambda\,u\,du + \int_{\pi}^{\infty} f(u) \sin \lambda\,u\,du\right\} d\lambda$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \sin \lambda\,x \left\{\int_{0}^{\pi} \sin \lambda\,u\,du + \int_{\pi}^{\infty} (0) \sin \lambda\,u\,du\right\} d\lambda, \quad \text{using (1)}$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \sin \lambda\,x \left[-\frac{\cos \lambda\,u}{\lambda}\right]_{0}^{\pi} d\lambda = \frac{2}{\pi}\int_{0}^{\infty} \frac{1 - \cos \lambda\,\pi}{\lambda} \sin \lambda\,x\,d\lambda.$$

Hence

$$\int_{0}^{\infty} \frac{1 - \cos \lambda\,\pi}{\lambda} \sin \lambda\,x\,d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2, & \text{for } 0 \le x \le \pi \\ 0, & \text{for } x > \pi \end{cases} \quad \text{[Using the given value of f(x)]}$$

At $x = \pi$, which is a point of discontinuity of f(x), the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi - 0) + (\pi + 0)}{2}\right] = \frac{\pi}{2} \times \frac{1 + 0}{2} = \frac{\pi}{4}.$$

Exercise :

1. Using Fourier cosine integral formula, show that $e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos ux}{u^2+a^2}\, dx, a > 0, x \geq 0.$

2. Show that $e^{-x} - e^{-2x} = \frac{6}{\pi}\int_0^{\infty} \frac{u \sin ux\, du}{(u^2+1)(u^2+4)}, x \geq 0.$

3. Suppose a function $f(x)$ is given by

$$f(x) = \begin{cases} 0, & \text{when } x \leq 0 \text{ and } x \geq \pi \\ \cos x, & \text{when } 0 \leq x \leq \pi \end{cases}$$

Using Fourier integral formula, show that

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{u[\sin u\, x - \sin u\, (\pi - x)]}{u^2 - 1}\, du.$$

4. Find a Fourier cosine and Fourier sine integral representation of the function

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

Ans. $f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos(\pi s/2)}{1-s^2} \cos s\, x\, ds; \quad f(x) = \frac{2}{\pi}\int_0^{\infty} \left(\frac{s - \sin(\pi s/2)}{s^2-1}\right) \sin s\, x\, dx$

5. If $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

(a) show that $f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos sx + s \sin sx}{s^2+1}\, ds$

(b) Verify directly that the above integral representation converges to the value $1/2$ at $x = 0$.

Fourier transform or Complex Fourier Transform:

Definition. Let $F(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty, \infty)$. Then the Fourier transform of $F(x)$ is a function of a new variable s and it is denoted and defined as

$$F\{F(x)\} = \bar{F}(s) = f(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx \quad \ldots (1)$$

The function $F(x)$ is then called inverse Fourier transform of $\bar{F}(s)$ or $f(s)$ and is denoted by

$$F(x) = F^{-1}\{\bar{F}(s)\} \quad \text{or} \quad F(x) = F^{-1}\{f(s)\} \quad \ldots (2)$$

Note: Some authors use p in place of s in (1) and (2). Also, they use $f(x)$ and $F(s)$ in place of $F(x)$ and $f(s)$ respectively.

Inversion formula for Fourier Transform or Complex Fourier Transform:

If $\bar{F}(s)$ is the Fourier transform of $F(x)$ and if $F(x)$ satisfies the Dirichlet conditions in every finite interval $(-l, l)$ and further if $F(x)$ is absolutely integrable in $(-\infty, \infty)$, then at every point of continuity of $F(x)$,

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s) e^{-isx}\, ds.$$

Proof: Complex Fourier integral formula is given by

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} F(u) e^{isu}\, du\right\} ds. \quad \ldots (3)$$

Complex Fourier transform of $F(x)$ is given by

$$\bar{F}(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx = \int_{-\infty}^{\infty} F(u) e^{isu}\, du. \quad \ldots (4)$$

Then, (3) and (4) $\Rightarrow$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s) e^{-isx}\, ds, \quad \ldots (5)$$

which is the required inversion formula for complex Fourier transform.

Thus, we have

$$F\{F(x)\} = \bar{F}(s) = f(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx \quad \ldots 1(a)$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \bar{F}(s)\, ds \quad \text{or} \quad F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds \quad \ldots 1(b)$$

(i)
$$F\{F(x)\} = \bar{F}(s) = f(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} F(x)\, dx \quad \ldots 2(a)$$

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} \bar{F}(s)\, ds \quad \text{or} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds \quad \ldots 2(b)$$

It is also known as symmetric form.

(ii)
$$F\{F(x)\} = \bar{F}\{F(s)\} = f(s) = \int_{-\infty}^{\infty} e^{-isx} F(x)\, dx \quad \ldots 3(a)$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} \bar{F}(s)\, ds \quad \text{or} \quad F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} f(s)\, ds \quad \ldots 3(b)$$

(iii) $F\{F(x)\} = \bar{F}(s) = f(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} F(x)\, dx.$ ... 4(a)

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, \bar{F}(s)\, ds \quad \text{or} \quad F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, f(s)\, ds \quad \text{... 4(b)}$$

It is also known as symmetric form.

Theorem. If $f(s)$ is Fourier transform of $F(x)$, then

- (i) $F\{F(-x)\} = f(-s)$
- (ii) $F\left\{\overline{F(-x)}\right\} = \overline{f(s)}$
- (iii) $F\left\{\overline{F(x)}\right\} = \overline{f(-s)}$,
  where bar over a quantity represents its complex conjugate.

Proof: By definition,

$$F\{F(x)\} = f(s) = \int_{-\infty}^{\infty} F(x) e^{isx}\, dx. \quad \text{... (1)}$$

(i) $F\{F(-x)\} = \int_{-\infty}^{\infty} F(-x) e^{isx}\, dx = \int_{\infty}^{-\infty} F(y) e^{-isy}(-dy)$, putting $y = -x$ and $dx = -dy$

$$= \int_{-\infty}^{\infty} F(x) e^{i(-s)x}\, dx = f(-s), \text{ by (1)}$$

(ii) $F\left\{\overline{F(-x)}\right\} = \int_{-\infty}^{\infty} \left\{\overline{F(-x)}\right\} e^{isx}\, dx = \int_{\infty}^{-\infty} \overline{F(y)}\, e^{-isy}(-dy)$, putting $y = -x$ and $dx = -dy$

$$= \int_{-\infty}^{\infty} \overline{F(x)}\, e^{-isx}\, dx = \left\{\int_{-\infty}^{\infty} F(x) e^{isx}\, dx\right\} = \overline{f(s)}, \text{ by (1)}$$

Fourier sine transform or infinite Fourier sine transform.

Definition. Let $F(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty, \infty)$. Then the infinite Fourier sine transform of $F(x)$ is a function of a new variables s and it is denoted and defined as

$$F_s\{F(x)\} = \bar{F}_s(s) = f_s(s) = \int_0^{\infty} F(x) \sin s\, x\, dx \quad \text{... (1)}$$

The function $F(x)$ is then called inverse Fourier sine transform of $\bar{F}_s(s)$ or $f_s(s)$ and is denoted by

$$F(x) = F_s^{-1}\{\bar{F}_s(s)\} \quad \text{or} \quad F(x) = F_s^{-1}\{f_s(s)\} \quad \text{... (2)}$$

Note: Some authors use p in place of s in (1) and (2).

Inversion formula for Inverse Fourier sine transform : If $\bar{F}_s(s)$ is the infinite Fourier sine transform of $F(x)$ and if $F(x)$ satisfies the Dirichlet conditions in every finite interval $(-l, l)$ and further if $F(x)$ is absolutely integrable in $(-\infty, \infty)$, then at every point of continuity of $F(x)$,

$$F(x) = \frac{2}{\pi}\int_0^{\infty} \bar{F}_s(s) \sin s x \, ds$$

Proof. Fourier sine integral formula is given by

$$F(x) = \frac{2}{\pi}\int_0^{\infty} \sin s x \left\{\int_0^{\infty} F(u) \sin s u \, du\right\} ds \quad \ldots (3)$$

Now, infinite Fourier sine transform of $F(x)$ is given by

$$\bar{F}_s(s) = \int_0^{\infty} F(x) \sin s x \, dx = \int_0^{\infty} F(u) \sin s u \, du \quad \ldots (4)$$

Then, (3) and (4) $\Rightarrow$

$$F(x) = \frac{2}{\pi}\int_0^{\infty} \bar{F}_s(s) \sin s x \, ds, \quad \ldots (5)$$

which is the required inversion formula for infinite Fourier sine transform.

Thus, we have

$$F_s\{F(x)\} = \bar{F}_s(s) = f_s(s) = \int_0^{\infty} F(x) \sin s x \, dx \quad \ldots 1(a)$$

$$\text{and} \quad F(x) = F_s^{-1}\{\bar{F}_s(s)\} = \frac{2}{\pi}\int_0^{\infty} \bar{F}_s(s) \sin s x \, ds \quad \ldots 1(b)$$

Some authors define $1(a)$ and $1(b)$ in the following symmetric form.

$$F_s\{F(x)\} = \bar{F}_s(s) = f_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(x) \sin s x \, dx \quad \ldots 2(a)$$

$$\text{and} \quad F(x) = F_s^{-1}\{\bar{F}_s(s)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{F}_s(s) \sin s x \, ds \quad \ldots 2(b)$$

Fourier cosine transform or infinite Fourier cosine transform.

Definition. Let $F(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty, \infty)$. Then the infinite Fourier cosine transform of $F(x)$ is a function of a new variable s and it is denoted and defined as...

$$F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \int_0^{\infty} F(x) \cos s x \, dx \quad \ldots (1)$$

The function $F(x)$ is then called inverse Fourier cosine transform of $\bar{F}_c(s)$ or $f_c(s)$ and is denoted by

$$F(x) = F_c^{-1}\{\bar{F}_c(s)\} \quad \text{or} \quad F(x) = F_c^{-1}\{f_c(s)\} \quad \ldots(2)$$

Inversion formula for inverse Fourier cosine transform : If $\bar{F}_c(s)$ is the infinite Fourier cosine transform of $F(x)$ and if $F(x)$ satisfies the Dirichlet conditions in every finite interval $(-l, l)$ and further if $F(x)$ is absolutely integrable in $(-\infty, \infty)$, then at every point of continuity of $F(x)$,

$$F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos s\,x\,ds.$$

Proof: Fourier cosine integral formula is given by

$$F(x) = \frac{2}{\pi}\int_0^\infty \cos s\,x\left\{\int_0^\infty F(u)\cos s\,u\,du\right\}ds \quad \ldots(3)$$

Now, infinite Fourier cosine transform of $F(x)$ is given by

$$\bar{F}_c(s) = \int_0^\infty F(x)\cos s\,x\,dx = \int_0^\infty F(u)\cos s\,u\,du \quad \ldots(4)$$

Then, (3) and (4) $\Longrightarrow$

$$F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos s\,x\,ds, \quad \ldots(5)$$

which is the required inversion formula for infinite Fourier cosine transform.

Thus, we have

$$F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \int_0^\infty F(x)\cos s\,x\,dx \quad \ldots 1(a)$$

and

$$F(x) = F_c^{-1}\{\bar{F}_c(s)\} = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos s\,x\,ds \quad \ldots 1(b)$$

Some authors define $1(a)$ and $1(b)$ in the following symmetric form

$$F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos s\,x\,dx \quad \ldots 2(a)$$

and

$$F(x) = F_c^{-1}\{\bar{F}_c(s)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{F}_c(s)\cos s\,x\,dx \quad \ldots 2(b)$$

Relationship between Fourier and Laplace transforms.

Let

$$F(t) = \begin{cases} e^{-xt}G(t), & t > 0 \\ 0, & t < 0. \end{cases}$$

Then,

$$F\{F(t)\} = \int_{-\infty}^{\infty} e^{ist}\, F(t)dt, \text{ by definition of Fourier transform}$$

$$= \int_{-\infty}^{0} e^{ist}\, F(t)dt + \int_{0}^{\infty} e^{ist}\, F(t)dt = \int_{0}^{\infty} e^{ist}\, e^{-xt}G(t)dt = \int_{0}^{\infty} e^{-(x-is)t}\, G(t)dt, \text{ by (1)}$$

Hence, $F\{F(t)\} = \int_0^{\infty} e^{-pt}\, G(t)dt$, taking $x - is = p$

$$= L\{G(t)\}, \text{ by the definition of Laplace transform}$$

Thus,

$$F\{F(t)\} = L\{G(t)\},$$

which is the required relationship between Fourier and Laplace transforms.

Some useful results of Integral Calculus for direct applications.

1. $\int e^{ax} \sin b\,x\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin b\,x - b \cos b\,x)$, $\int e^{ax} \cos b\,x\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos b\,x + b \sin b\,x)$

$$\int_0^{\infty} e^{-ax} \sin b\,x\,dx = \frac{b}{a^2+b^2} \quad \text{and} \quad \int_0^{\infty} e^{-ax} \cos b\,x\,dx = \frac{a}{a^2+b^2}$$

2. $\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right) = \frac{(-1)^n \times n!}{(a^2+x^2)^{(n+1)/2}} \cos\{(n+1)\tan^{-1}(a/x)\}$,

$$\frac{d^n}{dx^n}\left(\frac{a}{a^2+x^2}\right) = \frac{(-1)^n \times n!}{(a^2+x^2)^{(n+1)/2}} \sin\{(n+1)\tan^{-1}(a/x)\}.$$

3. $\int_0^{\infty} \frac{\sin px}{x} dx = \begin{cases} \pi/2, & \text{if } p > 0 \\ -\pi/2, & \text{if } p < 0. \end{cases}$

4. $\int_0^{\infty} e^{-x^2}\, dx = \sqrt{\pi}, \quad \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

5. $\int_0^{\infty} \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}} dx = \frac{1}{2}\tan\frac{a}{2}, \quad \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} dx = \frac{1}{2}\sec\frac{a}{2}$

6. $\int_0^{\infty} \frac{\cos px}{1+p^2} dp = \frac{\pi}{2}e^{-x}, \quad \int_0^{\infty} \frac{p \sin px}{1+p^2} dp = \frac{\pi}{2}e^{-x}$

1.11. Linearity property of Fourier transforms:

If c_1 and c_2 be constants, then

- (i) $F\{c_1F_1(x) + c_2F_2(x)\} = c_1F\{F_1(x)\} + c_2F\{F_2(x)\}$
- (ii) $F_s\{c_1F_1(x) + c_2F_2(x)\} = c_1F_s\{F_1(x)\} + c_2F_s\{F_2(x)\}$
- (iii) $F_c\{c_1F_1(x) + c_2F_2(x)\} = c_1F_c\{F_1(x)\} + c_2F_c\{F_2(x)\}$

Proof (i) By definition

$$F\{c_1F_1(x) + c_2F_2(x)\} = \int_{-\infty}^{\infty} e^{isx}\{c_1F_1(x) + c_2F_2(x)\}dx$$

$$= c_1\int_{-\infty}^{\infty} e^{isx}F_1(x)dx + c_2\int_{-\infty}^{\infty} e^{isx}F_2(x)dx = c_1F\{F_1(x)\} + c_2F\{F_2(x)\}, \text{ by definition}$$

(ii) By definition

$$F_s\{c_1F_1(x) + c_2F_2(x)\} = \int_0^{\infty} \{c_1F_1(x) + c_2F_2(x)\}\sin s\,x\,dx$$

$$= c_1\int_0^{\infty} F_1(x)\sin s\,x\,dx + c_2\int_0^{\infty} F_2(x)\sin s\,x\,dx$$

$$= c_1F_s\{F_1(x)\} + c_2F_s\{F_2(x)\}, \text{ by definition}$$

Change of scale property:

- (i) If $f(s)$ is the Fourier transform of $F(x)$, then $(1/|a|)f(s/a)$ is the Fourier transform of $F(ax)$, where $a \neq 0$.

  If $F\{F(x)\} = f(s)$, then $F\{F(ax)\} = (1/|a|)f(s/a)$, where $a \neq 0$.
- (ii) If $F_s\{F(x)\} = f_s(s)$, then $F_s\{F(ax)\} = (1/a)f_s(s/a)$.
- (iii) If $F_c\{F(x)\} = f_c(s)$, then $F_c\{F(ax)\} = (1/a)f_c(s/a)$.

Proof: (i) The following two cases arise :

Case I. Let $a > 0$. Then, by definition of Fourier transform, we get

$$F\{F(ax)\} = \int_{-\infty}^{\infty} e^{isx}F(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t}F(t)dt$$

[Putting $ax = t$ so that $dx = (1/a)dt$]

$= (1/a)f(s/a)$, by definition of Fourier transform.

Case II. Let $a < 0$. Let $b > 0$ such that $a = -b$. Then, by definition, we have

$$F\{F(ax)\} = \int_{-\infty}^{\infty} e^{isx}F(ax)dx = \int_{-\infty}^{\infty} e^{isx}F(-bx)dx, \text{ as } a = -b$$

$$= \int_{\infty}^{-\infty} e^{i(-s/b)t}F(t)\frac{dt}{(-b)}, \text{ putting } -bx = t \text{ and } dx = -(1/b)dt$$

$$= \frac{1}{b}\int_{-\infty}^{\infty} e^{i(s/a)t}\, F(t)dt = \frac{1}{(-a)} f(s/a), \text{ by definition}$$

Combining the above two cases,

$$F\{F(ax)\} = (1/|a|)f(s/a)$$

(ii) By definition

$$\int_0^{\infty} F(x) \sin s\,x\, dx = f_s(s). \quad \ldots (1)$$

Now, $F_s\{F(ax)\} = \int_0^{\infty} F(ax) \sin s\,x\, dx$, by definition

$$= \frac{1}{a}\int_0^{\infty} F(t) \sin \frac{st}{a} dt, \text{ putting } ax = t \text{ so that } dx = \frac{dt}{a}$$

$$= \frac{1}{a}\int_0^{\infty} F(x) \sin\{(s/a) \times x\}\, dx = \frac{1}{a} f_s(s/a), \text{ on replacing s by } s/a \text{ in } (1)$$

Shifting property for Fourier Transform.

If $f(s)$ is the complex Fourier transform of $F(x)$, then complex Fourier transform of $F(x - a)$ is $e^{isa}f(s)$, i.e. if $F\{F(x)\} = f(s)$, then $F\{F(x - a)\} = e^{isa}f(s)$.

Proof: By definition

$$\int_{-\infty}^{\infty} e^{isx}\, F(x)dx = f(s). \quad \ldots (1)$$

Now, $F\{F(x - a)\} = \int_{-\infty}^{\infty} e^{isx}\, F(x - a)dx = \int_{-\infty}^{\infty} e^{is(t+a)}\, F(t)dt$, putting $x - a = t$ and $dx = dt$

$$= e^{isa}\int_{-\infty}^{\infty} e^{ist}\, F(t)dt = e^{isa}\int_{-\infty}^{\infty} e^{isx}\, F(x)dx = e^{isa} f(s), \text{ using } (1).$$

Modulation theorem

(i) If $f(s)$ is the Fourier transform of the function $F(x)$, then $[f(s + a) + f(s - a)]/2$ is the Fourier transform of $F(x) \cos a\,x$.

i.e., If $F\{F(x)\} = f(s)$, then $F\{F(x) \cos a\,x\} = [f(s + a) + f(s - a)]/2$

(ii) If $F_s\{F(x)\} = f_s(s)$, then $F_s\{F(x) \cos a\,x\} = [f_s(s + a) + f_s(s - a)]/2$

(iii) If $F_s\{F(x)\} = f_s(s)$, then $F_c\{F(x) \sin a\,x\} = [f_s(s + a) - f_s(s - a)]/2$

(iv) If $F_c\{F(x)\} = f_c(s)$, then $F_s\{F(x) \sin a\,x\} = [f_c(s - a) - f_c(s + a)]/2$

Proof. (i) By definition

$$\int_{-\infty}^{\infty} e^{isx} F(x)dx = f(s). \quad \dots (1)$$

Now, $F\{F(x)\cos a\,x\} = \int_{-\infty}^{\infty} e^{isx} F(x)\cos a\,x\,dx$, by definition 1.6

$$= \int_{-\infty}^{\infty} e^{isx} F(x)\frac{e^{iax}+e^{-iax}}{2}dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s+a)x} F(x)dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s-a)x} F(x)dx$$

$$= (1/2)\times f(s+a) + (1/2)\times f(s-a), \text{ using (1)}$$

$$= [f(s+a)+f(s-a)]/2$$

(ii) By definition

$$\int_{0}^{\infty} F(x)\sin s\,x\,dx = f_s(s). \quad \dots (1)$$

Now, $F_s\{F(x)\cos a\,x\} = \int_0^{\infty} F(x)\cos a\,x\sin s\,x\,dx$, by definition

$$= \frac{1}{2}\int_0^{\infty} F(x)(2\sin a\,x\cos a\,x)dx = \frac{1}{2}\int_0^{\infty} F(x)[\sin(s+a)\,x + \sin(s-a)\,x]dx$$

$$= \frac{1}{2}\left[\int_0^{\infty} F(x)\sin(s+a)\,x\,dx + \int_0^{\infty} F(x)\sin(s-a)\,x\,dx\right]$$

$$= \frac{1}{2}[f_s(s+a)+f_s(s-a)], \text{ using (1).}$$

Example: . Find the Fourier complex transform of f(x) if

$$f(x) = \begin{cases} e^{iwx}, & a < x < b \\ 0, & x < a \text{ or } x > b \end{cases}$$

Solution:

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)dx = \int_{-\infty}^{a} e^{isx} f(x)dx + \int_a^b e^{isx} f(x)dx + \int_b^{\infty} e^{isx} f(x)dx$$

$$= 0 + \int_a^b e^{isx}\, e^{iwx}dx + 0, \text{ using the given values of } f(x)$$

$$= \int_a^b e^{i(s+w)x}\,dx = \left[\frac{e^{i(s+w)x}}{i(s+w)}\right]_a^b = i\frac{e^{i(s+w)b} - e^{i(s+w)a}}{i^2(s+w)} = \frac{i}{s+w}\left[e^{i(s+w)a} - e^{i(s+w)b}\right], \text{ as } i^2$$

$$= -1.$$

Example: Find the Fourier transform of

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a. \end{cases}$$

Solution: By definition

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{-a} f(x)e^{isx}dx + \int_{-a}^{a} f(x)e^{isx}dx + \int_{a}^{\infty} f(x)e^{isx}dx$$

$$= \int_{-a}^{a} x\, e^{isx}dx, \text{ using the given values of } f(x)$$

$$= \left[x\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} \frac{e^{isx}}{is}dx, \text{ integrating by parts}$$

$$= \frac{a}{is}e^{isa} - \frac{(-a)}{is}e^{-isa} - \left[\frac{e^{isx}}{(is)^2}\right]_{-a}^{a} = \frac{a}{is}\left(e^{isa} + e^{-isa}\right) + \frac{1}{s^2}\left(e^{isa} - e^{-isa}\right)$$

$$= -(ia/s) \times (2\cos s\, a) + (1/s^2) \times (2i \sin s\, a) = (2i/s^2) \times (\sin sa - as\cos as).$$

Example: Find the Fourier transform of F(x), if

$$F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a. \end{cases}$$

Solution: By definition

$$F\{F(x)\} = \int_{-\infty}^{\infty} F(x)e^{isx}dx = \int_{-\infty}^{-a} F(x)e^{isx}dx + \int_{-a}^{a} F(x)e^{isx}dx + \int_{a}^{\infty} F(x)e^{isx}dx$$

$$= \int_{-a}^{a} x^2\, e^{isx}dx, \text{ using the given values of } F(x)$$

$$= \left[x^2\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} (2x)\frac{e^{isx}}{is}dx, \text{ integrating by parts}$$

$$= \frac{a^2}{is}e^{isa} - \frac{a^2}{is}e^{-isa} - \frac{2}{is}\left\{\left[x\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} 1 \cdot \frac{e^{isx}}{is}dx\right\}$$

$$= \frac{a^2}{is}\left(e^{isa} - e^{-isa}\right) - \frac{2}{is}\left\{\frac{a}{is}e^{isa} + \frac{a}{is}e^{-isa} - \left[\frac{e^{isx}}{(is)^2}\right]_{-a}^{a}\right\}$$

$$= (a^2/is) \times (2i \sin s\, a) - (2a/i^2s^2) \times \left(e^{isa} + e^{-isa}\right) + (2/i^3s^3) \times \left(e^{isa} - e^{-isa}\right)$$

$$= (2a^2/s)\sin s\, a + (2a/s^2)(2\cos s\, a) - (2/is^3)(2i \sin s\, a)$$

$$= (4a/s^2)\cos s\, a + (2/s^3)(a^2s^2 - 2)\sin s\, a.$$

Example: Find the Fourier complex transform of $F(x) = e^{-x|a|}$, where $a > 0$ and x belongs to $(-\infty, \infty)$.

Solution: By definition of the Fourier Complex transform, we have

$$F\{F(x)\} = \int_{-\infty}^{\infty} e^{isx} F(x)dx = \int_{-\infty}^{\infty} e^{isx} e^{-|a|x}dx$$

$$= \int_{-\infty}^{0} e^{isx} e^{ax}dx + \int_{0}^{\infty} e^{isx} e^{-ax}dx, \text{ noting that } |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{(a+is)x}\, dx + \int_{0}^{\infty} e^{-(a-is)x}\, dx = \left[\frac{e^{(a+is)x}}{a+is}\right]_{-\infty}^{0} + \left[\frac{e^{-(a-is)x}}{-(a-is)}\right]_{0}^{\infty}$$

$$= 1/(a+is) + 1/(a-is) = 2a/(a^2+s^2)$$

Example: Find the sine transform of a function of x which is equal to $\sin x$ for $0 < x < a$ and 0 for $x > a$.

Solution: Let the given function of x be $F(x)$, Then, we have

$$F(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a. \end{cases} \quad \ldots(1)$$

Then, $F_s\{F(x)\} = \int_0^{\infty} F(x) \sin s\,x\, dx$, by definition

$$= \int_0^{a} F(x) \sin s\,x\, dx + \int_a^{\infty} F(x) \sin s\,x\, dx = \int_0^{a} \sin x \sin s\,x\, dx + 0, \text{ using } (1)$$

$$\frac{1}{2}\int_0^{a} [\cos(s-1)x - \cos(s+1)x]\, dx = \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^{a}$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$$

Example: Find the cosine transform of a function of x which is unity for $0 < x < a$ and 0 for $x \geq a$. What is the function whose cosine transform is $(\sin a\,s)/s$?

Solution: Let the given function of x be $F(x)$. Then, we have

$$F(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x \geq a. \end{cases} \quad \ldots(1)$$

$$\therefore F_c\{F(x)\} = \int_0^{\infty} F(x) \cos s\,x\, dx = \int_0^{a} F(x) \cos s\,x\, dx + \int_a^{\infty} F(x) \cos s\,x\, dx$$

$$= \int_0^{a} \cos s\,x\, dx + 0, \text{ by } (1)$$

$$= \left[\frac{\sin s\,x}{s}\right]_0^{a} = \frac{\sin s\,a}{s}$$

Let $f(x)$ be the function such that $F_c\{f(x)\} = (\sin a\,s)/s$.

Hence by the definition of inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\,s}{s}\cos s\,x\,dx = \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)\,s + \sin(a-x)\,s}{s}\,dx$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\sin(a+x)\,s}{s}\,ds + \int_0^\infty \frac{\sin(a-x)\,s}{s}\,ds\right]$$

$$= \begin{cases}(1/\pi)\times(\pi/2+\pi/2), & x < a\\ (1/\pi)\times(\pi/2-\pi/2), & x > a\end{cases} = \begin{cases}1, & x < a\\ 0, & x > a'\end{cases}$$

Example: Find the Fourier sine and cosine transform of f(x), if

$$f(x) = \begin{cases} x, & 0 < x < 1\\ 2-x, & 1 < x < 2\\ 0, & x > 2.\end{cases}$$

Solution: By definition of Fourier sine transform, we have

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin s\,x\,dx = \int_0^1 f(x)\sin s\,x\,dx + \int_1^2 f(x)\sin s\,x\,dx + \int_2^\infty f(x)\sin s\,x\,dx$$

$$= \int_0^1 x\sin s\,x\,dx + \int_1^2 (2-x)\sin s\,x\,dx + 0, \text{ by given values of f(x)}$$

$$= \left[(x)\left(-\frac{\cos s\,x}{s}\right)\right]_0^1 - \int_0^1 1\times\left(-\frac{\cos s\,x}{s}\right)dx + \left[(2-x)\left(-\frac{\cos s\,x}{s}\right)\right]_1^2$$

$$- \int_1^2 (-1)\left(-\frac{\cos s\,x}{s}\right)dx \text{ on integrating by parts}$$

$$= -\frac{\cos s}{s} + \frac{1}{s}\left[\frac{\sin s\,x}{s}\right]_0^1 + \frac{\cos s}{s} - \frac{1}{s}\left[\frac{\sin s\,x}{s}\right]_1^2 = \frac{\sin s}{s^2} - \frac{\sin 2\,s - \sin s}{s^2}$$

$$= (1/s^2)\times(2\sin s - 2\sin s\cos s) = (2/s^2)\times\sin s\ (1-\cos s)$$

By definition of Fourier cosine transform, we have

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos s\,x\,dx = \int_0^1 f(x)\cos s\,x\,dx + \int_1^2 f(x)\cos s\,x\,dx + \int_2^\infty f(x)\cos s\,x\,dx$$

$$= \int_0^1 x\cos s\,x\,dx + \int_1^2 (2-x)\cos s\,x\,dx + 0, \text{ by given values of f(x)}$$

$$= \left[(x)\left(\frac{\sin s\,x}{s}\right)\right]_0^1 - \int_0^1 1\times\left(\frac{\sin s\,x}{s}\right)dx + \left[(2-x)\left(\frac{\sin s\,x}{s}\right)\right]_1^2 - \int_1^2 (-1)\left(\frac{\sin s\,x}{s}\right)dx$$

$$= \frac{\sin s}{s} - \left[-\frac{\cos s\,x}{s^2}\right]_0^1 - \frac{\sin s}{s} + \left[-\frac{\cos s\,x}{s^2}\right]_1^2 = \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2\,s}{s^2} + \frac{\cos s}{s^2}$$

$$= (1/s^2)\times\{2\cos s - (1+\cos 2\,s)\} = (1/s^2)\times(2\cos s - 2\cos^2 s)$$

$$= (2/s^2)\times\cos s\ (1-\cos s)$$

Example: Find the Fourier sine transform of e^{-x} and hence prove that

$$\int_0^\infty \frac{t \sin t\, x}{1+t^2} dt = \frac{\pi e^{-x}}{2}$$

Solution: By definition of Fourier sine transform of e^{-x}, we have

$$F_s\{e^{-x}\} = \int_0^\infty e^{-x} \sin s\, x\, dx = \frac{s}{s^2+1}, \quad \text{as} \quad \int_0^\infty e^{-ax} \sin b\, x\, dx = \frac{b}{a^2+b^2}$$

Then by inversion formula for Fourier sine transform, we get

$$e^{-x} = \frac{2}{\pi}\int_0^\infty F_s\{e^{-x}\} \sin s\, x\, ds = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \sin s\, x\, ds$$

$$\text{or} \quad \frac{2}{\pi}\int_0^\infty \frac{t \sin t\, x}{1+t^2} dt = e^{-x} \quad \text{so that} \quad \int_0^\infty \frac{t \sin t\, x}{1+t^2} dt = \frac{\pi}{2} e^{-x}.$$

Example:

(a) Find the Fourier cosine transform of e^{-x^2}

(b) Find the Fourier sine transform of $xe^{-x^2/2}$

Solution: (a) By definition,

$$F_c\{e^{-x^2}\} = \int_0^\infty e^{-x^2} \cos s\, x\, dx = I, \text{ say} \quad \ldots(1)$$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}(e^{-x^2} \cos s\, x) dx = \int_0^\infty e^{-x^2} (-x \sin s\, x) dx$$

$$\text{or} \quad \frac{dI}{ds} = \frac{1}{2}\int_0^\infty (\sin s\, x)\left(-2xe^{-x^2}\right) dx. \quad \ldots(2)$$

Putting $-x^2 = u$ so that $-2xdx = du$, we have

$$\int \left(-2xe^{-x^2}\right) dx = \int e^u\, du = e^u = e^{-x^2}. \quad \ldots(3)$$

Integrating by parts with $\left(-2x\, e^{-x^2}\right)$ as the second function and using (3), (2) reduces to

$$\frac{dI}{ds} = \frac{1}{2}\left\{\left[\sin s\, x\ e^{-x^2}\right]_0^\infty - \int_0^\infty s \cos s\, x.\, e^{-x^2} dx\right\} = -\frac{s}{2}\int_0^\infty e^{-x^2} \cos s\, x\, dx = -\frac{sI}{2}, \text{ using } (1).$$

Separating variables,

$$(1/I)\, dI = -(s/2)\, ds \quad \ldots(4)$$

Integrating, $\log I = -(s^2/4) + \log C \quad \text{or} \quad I = Ce^{-s^2/4}, \quad \ldots(4)$

where C is constant of integration.

Now, (1) and (4) ⇒

$$Ce^{-s^2/4} = \int_0^\infty e^{-x^2} \cos s\,x\,dx. \quad \ldots(5)$$

Putting $s = 0$ in (5), we get

$$C = \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2 \quad \ldots(6)$$

∴ (1), (4) and (5) ⇒

$$F_c\{e^{-x^2}\} = 1 = (\sqrt{\pi}/2)e^{-s^2/4}.$$

(b) We have, $F_s\{xe^{-x^2/2}\} = \int_0^\infty x\,e^{-x^2/2} \sin s\,x\,dx$, by definition

$$\text{or } F_s\{xe^{-x^2/2}\} = -\int_0^\infty \sin s\,x \times (-xe^{-x^2/2})\,dx. \quad \ldots(1)$$

Putting $-x^2/2 = u$ so that $-x\,dx = du$, we have

$$\int (-x)\,e^{-x^2/2}\,dx = \int e^u\,du = e^u = e^{-x^2/2} \quad \ldots(2)$$

Integrating by parts with $(-xe^{-x^2/2})$ as the second function and using (2), (1) reduces to

$$F_s\{xe^{-x^2/2}\} = -\left\{\left[\sin s\,x\,xe^{-x^2/2}\right]_0^\infty - \int_0^\infty s\cos s\,x\,e^{-x^2/2}\,dx\right\}$$

$$\text{or } F_s\{xe^{-x^2/2}\} = s\int_0^\infty \cos s\,x\,e^{-x^2/2}\,dx = sI, \quad \ldots(3)$$

where

$$I = \int_0^\infty \cos s\,x\,e^{-x^2/2}\,dx. \quad \ldots(4)$$

From (4),

$$\frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}(\cos s\,x\,e^{-x^2/2})\,dx = \int_0^\infty (-x\sin s\,x)\,e^{-x^2/2}\,dx$$

$$\text{or } \frac{dI}{ds} = \int_0^\infty \sin s\,x(-xe^{-x^2/2})\,dx = \left[\sin s\,x\,xe^{-x^2/2}\right]_0^\infty - \int_0^\infty s\cos s\,x\,e^{-x^2/2}\,dx$$

$$\text{or } \frac{dI}{ds} = -s\int_0^\infty \cos s\,x\,e^{-x^2/2}\,dx = -sI, \text{ using (4)}$$

$$\therefore (1/I)\,dI = -s\,ds$$

$$\text{so that } \log I = -(s^2/2) + \log C$$

$$\therefore \log(I/C) = -(s^2/2) \text{ or } I = Ce^{-(s^2/2)} \quad \ldots(5)$$

From (4) and (5),

$$Ce^{-(s^2/2)} = \int_0^\infty \cos s\,x\, e^{-x^2/2}\,dx \quad \ldots(6)$$

Put $s = 0$ in (6), we get

$$C = \int_0^\infty e^{-x^2/2}\,dx \quad \ldots(7)$$

We know that

$$\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2 \quad \ldots(8)$$

Putting $x = v/\sqrt{2}$ and $dx = dv/\sqrt{2}$, (8) reduces to

$$\int_0^\infty e^{-v^2/2}\frac{dv}{\sqrt{2}} = \frac{\sqrt{\pi}}{2} \quad \text{so that} \quad \int_0^\infty e^{-x^2/2}\,dx = \frac{\sqrt{2}\sqrt{\pi}}{2}$$

Now, (7) and (9) $\Rightarrow C = \sqrt{2}\sqrt{\pi}/2$.

Then, (4), (5) and (7) $\Rightarrow I = \left(\sqrt{2}\sqrt{\pi}/2\right)e^{-s^2/2}$

$$\therefore (3) \Rightarrow F_s\left\{xe^{-x^2/2}\right\} = sI = \sqrt{(\pi/2)}\,se^{-s^2/2}$$

Example: Find Fourier sine transform of $(1/x) \times e^{-ax}$

Solution: By definition,

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^\infty \frac{e^{-ax}}{x}\sin s\,x\,dx = I, \text{ say.} \quad \ldots(1)$$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{x}(x\cos s\,x)\,dx = \int_0^\infty e^{-ax}\cos s\,x\,dx = \frac{a}{a^2+s^2}$$

Thus, $dI = \{a/(a^2+s^2)\}\,ds$

Integrating, $I = \tan^{-1}(s/a) + c$, c being an arbitrary constant

For $s = 0$, (1) and (2) give $c + \tan^{-1} 0 = 0$ so that $c = 0$.

$\therefore$ From (1) and (2), $I = \tan^{-1}(s/a)$.

or $F_s\{e^{-ax}/x\} = \tan^{-1}(s/a)$

Example: Find Fourier cosine transform of $1/(1+x^2)$. Hence find Fourier sine transform of $x/(1+x^2)$.

Solution: By definition,

$$F_c\left\{\frac{1}{1+x^2}\right\} = \int_0^\infty \frac{\cos s\,x}{1+x^2}\,dx = I, \text{ say} \quad \ldots(1)$$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{-x\sin s\,x\,dx}{1+x^2} = -\int_0^\infty \frac{x\sin s\,x}{1+x^2}\,dx \quad \ldots(2)$$

$$\text{or } \frac{dI}{ds} = -\int_0^\infty \frac{x^2 \sin s\,x\,dx}{x(1+x^2)} = -\int_0^\infty \frac{[(1+x^2)-1]\sin s\,x}{x(1+x^2)}\,dx$$

$$= -\int_0^\infty \frac{\sin s\,x}{x}\,dx + \int_0^\infty \frac{\sin s\,x}{x(1+x^2)}\,dx$$

$$\therefore \frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin s\,x}{x(1+x^2)}\,dx, \quad \text{as } \int_0^\infty \frac{\sin s\,x}{x}\,dx = \frac{\pi}{2} \quad \ldots(3)$$

From (3),

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos s\,x}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos s\,x}{1+x^2}\,dx = I$$

or $(d^2I/ds^2) - I = 0$ or $(D^2-1)I = 0$, where $D = d/ds$

whose solution is $I = Ae^{-s} + Be^{s}$, A and B being arbitrary constants ... (4)

From (4), $dI/ds = -Ae^{-s} + Be^{s}$ ... (5)

For $s = 0$, (1) and (4) give $A + B = \int_0^\infty \frac{dx}{1+x^2} = [\tan^{-1} x]_0^\infty = \frac{\pi}{2}$. ... (6)

For $s = 0$, (3) and (5) give $-A + B = \pi/2$. ... (7)

Solving (6) and (7), $A = \pi/2, B = 0$. So from (1) and (4), we have

$$I = (\pi/2) \times e^{-s} \quad \text{or} \quad F_c\{1/(1+x^2)\} = (\pi/2) \times e^{-s} \quad \ldots(8)$$

Second part. We have $F_s\left\{\frac{x}{1+x^2}\right\} = \int_0^\infty \frac{x \sin sx}{1+x^2}\,dx$, by definition

$= -dI/ds$, from (2) of part (a)

$= (\pi/2) \times e^{-s}$, from (5) with $A = \pi/2$ and $B = 0$.

Exercise :

1. Find the Fourier cosine transform of $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

Ans. $\{(1-s)\sin(1+s)\,a + (1+s)\sin(1-s)\,a\}/2(1-s^2)$

2. Find the Fourier sine and cosine transform of $F(x)$, if

$F(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$ **Ans.** $F_s\{F(x)\} = (1-\cos s)/s$; $F_c\{F(x)\} = (1/s) \times \sin s$

3. Find $F(x)$ if its cosine transform is given by

$\overline{F}_c(s) = \begin{cases} (1/2\pi) \times (a - s/2), & s < 2a \\ 0, & s \ge 2a \end{cases}$ **Ans.** $(1-\cos 2\,xa)/2\pi^2x^2$

4. Find the Fourier cosine transform of F(x), where $F(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x < b \\ 0, & x > b \end{cases}$

Ans. $(a\cos s\,a - b\cos s\,b)/s + (\sin s\,b - \sin s\,a)/s^2$

5. Find F(x) if $\overline{F}_c(s) = (\pi/2) \times e^{-s}$ **Ans.** $1/(1+s^2)$

6. Find the sine and cosine transform of $2e^{-5x} + 5e^{-2x}$.

Ans. $(2s)/(s^2+25) + (5s)/(s^2+4)$

7. Find the sine and cosine transform of $(e^{ax} + e^{-ax})/(e^{\pi x} - e^{-\pi x})$.

Ans. $(\sinh s)/2(\cos a + \cosh s)$; $\{\cos(a/2)\cosh(s/2)\}/(\cos a + \cosh s)$

8. Find the Fourier sine and cosine transform of x^{m-1}.

Ans. $F_s\{x^{m-1}\} = \{\Gamma(m)/s^m\} \times \sin(m\pi/2)$; $F_c\{x^{m-1}\} = \{\Gamma(m)/s^m\} \times \cos(m\pi/2)$

9. Find the cosine transform of $1/(x^2 + a^2)$. **Ans.** $(\pi/2) \times e^{-as}$

10. Find F(x) if its sine transform is $\pi/2$. **Ans.** $\pi/2x$

11. Find the sine transform of $1/(e^{\pi x} - e^{-\pi x})$ and deduce that

$$F_s(\text{cosech } \pi x) = \left(1/\sqrt{2\pi}\right) \times \tanh(s/2).$$

12. Find the Fourier sine transform of $F(x) = 1/x$. **Ans.** $\pi/2$

13. Find Fourier sine and cosine transform of e^{-x} and using the inversion formula recover the original functions in both the cases.

Ans. $F_s\{e^{-x}\} = s/(1+s^2)$; $F_c\{e^{-x}\} = 1/(1+s^2)$

14. Find the Fourier transform of f(x), if $f(x) = \begin{cases} \sqrt{2\pi}/2\varepsilon, & |x| < \varepsilon \\ 0, & |x| > \varepsilon. \end{cases}$

Ans. $(\sin s\ \varepsilon)/(s\,\varepsilon)$

15. Find f(x) if (i) its sine transform is e^{-as} (ii) its cosine transform is e^{-as}

Ans. (i) $F_s^{-1}[e^{-as}] = 2x/\pi(a^2 + x^2)$ (ii) $F_c^{-1}[e^{-as}] = 2a/\pi(a^2 + x^2)$

16. Find Fourier Complex Transform of $(\sin a\,x)/x$. **Ans**. π, if $|s| < a$; 0, if $|s| > a$.

17. Find (i) $F_c\{x^n\}$ (ii) $F_s\{x^n\}$.

Ans. (i) $[\pi s^{n-1}/2\Gamma(n)]\sec(n\pi/2)$ (ii) $[\pi s^{n-1}/2\Gamma(n)]\text{cosec}(n\pi/2)$

18. Show that $F\left\{\frac{\cosh ax}{\cosh \pi x}\right\} = \frac{2\cos(a/2)\cosh(s/2)}{\cos a+\cosh s}$, $-\pi < a < \pi$.

Example: Solve the integral equation

$$\int_0^\infty F(x)\cos s\,x\,dx = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$$

Hence deduce that

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

Solution: Let $\int_0^\infty F(x)\cos d\,x = f_c(s)$ ... (1)

where $f_c(s) = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1. \end{cases}$ ... (2)

Then, by definition, $f_c(s)$ is Fourier cosine transform of $F(x)$. In order to solve the given integral equation, we must obtain value of $F(x)$.

Now, by inversion formula for Fourier cosine transform, we get

$$F(x) = F_c^{-1}\{f_c(s)\} = \frac{2}{\pi}\int_0^\infty f_c(s)\cos s\,x\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 f_c(s)\cos s\,x\,ds + \int_1^\infty f_c(s)\cos s\,x\,ds\right] = \frac{2}{\pi}\left[\int_0^1 (1-s)\cos s\,x\,ds + 0\right], \text{ using (2)}$$

$$= \frac{2}{\pi}\left[\left[(1-s)\frac{\sin s\,x}{x}\right]_0^1 - \int_0^1 (-1)\frac{\sin s\,x}{x}\,ds\right] = \frac{2}{\pi}\left\{0 + \frac{1}{x}\left[-\frac{\cos s\,x}{x}\right]_0^1\right\}$$

$$= \frac{2}{\pi x^2}(1-\cos x). \quad \text{... (3)}$$

Deduction. From (1), (2) and (3), we have

$$\int_0^\infty \frac{2}{\pi x^2}(1-\cos x)\cos s\,x\,dx = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases} \quad \text{... (4)}$$

Taking limit as $s \to 0$ of both sides of (4), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = 1 \quad \text{or} \quad \frac{2}{\pi}\int_0^\infty \frac{2\sin^2(x/2)}{x^2}\,dx = 1 \quad \text{... (5)}$$

Putting $x = 2t$ so that $dx = 2dt$, (5) reduces to

$$\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 t}{4t^2}(2dt)=1 \quad \text{or} \quad \int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt=\frac{\pi}{2}.$$

Example: Solve the integral equation $\int_0^{\infty} f(x)\cos\lambda x\,dx = e^{-\lambda}$

Solution: Let $\int_0^{\infty} f(x)\cos\lambda x\,dx = f_c(\lambda), \quad \ldots(1)$

where $f_c(\lambda) = e^{-\lambda}. \quad \ldots(2)$

[Here λ has been used in place of the usual variable s]

Then, by definition, $f_c(\lambda)$ is Fourier cosine transform of $f(x)$. In order to solve the given integral equation, we must obtain value of $f(x)$.

Now, by inversion formula for Fourier cosine transform, we get

$$f(x) = F_c^{-1}\{f_c(\lambda)\} = \frac{2}{\pi}\int_0^{\infty} f_c(\lambda)\cos\lambda x\,d\lambda = \frac{2}{\pi}\int_0^{\infty} e^{-\lambda}\cos\lambda x\,d\lambda, \text{ using } (2)$$
$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + x\sin\lambda x)\right]_{\lambda=0}^{\infty} = \frac{2}{\pi(1+x^2)}.$$

Example: Solve for $F(x)$ the integral equation:

$$\int_0^{\infty} F(x)\sin x t\,dx = \begin{cases}1, & 0\le t<1\\ 2, & 1\le t\le 2\\ 0, & t\ge 2.\end{cases}$$

Solution: Let $\int_0^{\infty} F(x)\sin x t\,dx = f_s(t), \quad \ldots(1)$

where $f_s(t) = \begin{cases}1, & 0\le t<1\\ 2, & 1\le t\le 2\\ 0, & t\ge 2.\end{cases} \quad \ldots(2)$

[Here t has been used in place of the usual variable s]

Then, by definition, $f_s(t)$ is Fourier sine transform of $F(x)$.

Now, by inversion formula for Fourier sine transform, we have

$$F(x) = F_s^{-1}\{f_s(t)\} = \frac{2}{\pi}\int_0^{\infty} f_s(t)\sin t x\,dt$$
$$= \frac{2}{\pi}\left[\int_0^1 f_s(t)\sin t x\,dt + \int_1^2 f_s(t)\sin t x\,dt + \int_2^{\infty} f_s(t)\sin t x\,dt\right]$$
$$= \frac{2}{\pi}\left[\int_0^1 \sin t x\,dt + \int_1^2 (2\times\sin t x)\,dt + 0\right] = \frac{2}{\pi}\left\{\left[-\frac{\cos t x}{x}\right]_0^1 + 2\left[-\frac{\cos t x}{x}\right]_1^2\right\}$$
$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right] = \frac{2}{\pi x}(1+\cos x - 2\cos 2x).$$

Example: Show that $\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1}\,d\lambda = \frac{\pi}{2}e^{-x}, x \geq 0$

Solution: We know that $\int_0^\infty e^{-x}\cos\lambda\, x\, dx = \frac{1}{1+\lambda^2}$ ... (1)

Re-writing (1), $\int_0^\infty e^{-x}\cos\lambda\, x\, dx = f_c(\lambda)$, ... (2)

where $f_c(\lambda) = 1/(1+\lambda^2)$. ... (3)

[Here λ has been used in place of the usual variable s]

Then, by definition, $f_c(\lambda)$ is Fourier cosine transform of e^{-x}. So, by inversion formula for Fourier cosine transform, we have

$$e^{-x} = F_c^{-1}\{f_c(\lambda)\} = \frac{2}{\pi}\int_0^\infty f_c(\lambda)\cos\lambda\, x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda\, x}{1+\lambda^2}\,d\lambda, \text{ using (1)}$$

$$\therefore \int_0^\infty \frac{\cos\lambda\, x}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-x}.$$

Convolution or Falting. The convolution of two functions $F(x)$ and $G(x)$, where $-\infty < x < \infty$, is denoted and defined as

$$F * G = \int_{-\infty}^{\infty} F(u)G(x-u)\,du \quad \text{or} \quad F * G = \int_{-\infty}^{\infty} G(u)F(x-u)\,du.$$

The convolution or Falting theorem for Fourier transforms The Fourier transform of the convolution of $F(x)$ and $G(x)$ is the product of the Fourier transforms of $F(x)$ and $G(x)$, i.e.,

$$F\{F * G\} = F\{F(x)\} \cdot F\{G(x)\}.$$

Proof:

$$\text{L.H.S.} = F\left\{\int_{-\infty}^{\infty} F(u)G(x-u)\,du\right\}, \quad \text{by definition of convolution}$$

$$= \int_{-\infty}^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} F(u)G(x-u)\,du\right\}dx, \quad \text{by definition of Fourier transform}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{isx}\,G(x-u)\,dx\right\}du, \quad \text{changing the order of integration}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{is(u+v)}\,G(v)\,dv\right\}du, \quad \text{putting } x-u=v \text{ so that } dx=dv$$

$$= \int_{-\infty}^{\infty} e^{isu}\, F(u) \left\{ \int_{-\infty}^{\infty} e^{isv}\, G(v)\, dv \right\} du = \int_{-\infty}^{\infty} e^{isu}\, F(u) F\{G(x)\}\, du, \quad \text{by definition}$$

$$= F\{G(x)\} \int_{-\infty}^{\infty} e^{isu}\, F(u)\, du = F\{G(x)\} \cdot F\{F(x)\}, \quad \text{by definition again}$$

Parseval's identity for Fourier transform. Rayleigh's Theorem or Plancherel's theorem.

If $f(s)$ and $g(s)$ are the complex Fourier transforms of $F(x)$ and $G(x)$ respectively, then

$$\text{(i)} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \overline{g(s)}\, ds = \int_{-\infty}^{\infty} F(x) \overline{G(x)}\, dx$$

$$\text{(ii)} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|^2\, ds = \int_{-\infty}^{\infty} |F(x)|^2\, dx,$$

where bar signifies the complex conjugate.

Proof: (i) Using the inversion formula for Fourier transform, we get

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) e^{-isx}\, ds \quad \ldots (1)$$

Taking complex conjugates on both sides, (1) gives

$$\overline{G(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds \quad \ldots (2)$$

$$\therefore \int_{-\infty}^{\infty} F(x) \overline{G(x)}\, dx = \int_{-\infty}^{\infty} F(x) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds \right\} dx, \quad \text{using (2)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g(s)} \left\{ \int_{-\infty}^{\infty} F(x) e^{isx}\, dx \right\} ds, \quad \text{changing the order of integration}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g(s)}\, f(s)\, ds, \quad \text{by definition of Fourier transform}$$

(ii) Taking $G(x) = F(x)$ in part (i), we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \overline{f(s)}\, ds = \int_{-\infty}^{\infty} F(x) \overline{F(x)}\, dx \quad \text{or} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|^2\, ds = \int_{-\infty}^{\infty} |F(x)|^2\, dx.$$

Note. Parseval's identities for Fourier cosine and sine transforms as above the following:

$$\text{(i)} \quad \frac{2}{\pi} \int_{0}^{\infty} f_c(s) g_c(s)\, ds = \int_{0}^{\infty} F(x) G(x)\, dx.$$

(ii) $\frac{2}{\pi}\int_0^\infty f_s(s)g_s(s)\,ds = \int_0^\infty F(x)G(x)\,dx.$

(iii) $\frac{2}{\pi}\int_0^\infty [f_c(s)]^2\,ds = \int_0^\infty [F(x)]^2\,dx.$

(iv) $\frac{2}{\pi}\int_0^\infty [f_s(s)]^2\,ds = \int_0^\infty [F(x)]^2\,dx.$

Example: Find Fourier transform of $F(x)$ defined by $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence prove that

$$\int_0^\infty \frac{\sin^2 a x}{x^2}\,dx = \frac{\pi a}{2}$$

Solution: Given that

$$F(x) = \begin{cases} 1, & -a < x < a \\ 0, & x < -a \text{ or } x > a. \end{cases} \quad \ldots(1)$$

Let $f(s)$ be Fourier transform of $F(x)$. Then, by definition

$$f(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx = \int_{-\infty}^{-a} e^{isx} F(x)\,dx + \int_{-a}^{a} e^{isx} F(x)\,dx + \int_{a}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-a}^{a} e^{isx}\,dx, \text{ by } (1)$$

$$= \left[\frac{1}{is} e^{isx}\right]_{-a}^{a} = \left(\frac{1}{is}\right) \times \left(e^{isa} - e^{-isa}\right) = (2/s) \times \sin s\,a. \quad \ldots(2)$$

Using Parseval's identity for Fourier transform,

$$\int_{-\infty}^{\infty} [F(x)]^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)]^2\,ds$$

$$\text{or } \int_{-\infty}^{-a} [F(x)]^2\,dx + \int_{-a}^{a} [F(x)]^2\,dx + \int_{a}^{\infty} [F(x)]^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)]^2\,ds$$

$$\text{or } 0 + \int_{-a}^{a} (1^2)\,dx + 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2\sin s\,a}{s}\right)^2 ds, \quad \text{putting values of } F(x) \text{ and } f(s) \text{ from } (1) \text{ and } (2)$$

$$\text{or } 2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 a x}{x^2}\,dx = \frac{2}{\pi} \times 2\int_0^\infty \frac{\sin^2 a x}{x^2}\,dx, \quad \text{as } \frac{\sin^2 a x}{x^2} \text{ is an even function of } x$$

$$\text{Thus, } \int_0^\infty \frac{\sin^2 a x}{x^2}\,dx = \frac{\pi a}{2}.$$

Exercise:

1. If $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and $f(s) = \frac{2\sin sa}{s}, s \neq 0$, then

prove that $\int_0^\infty \frac{\sin^2 ax}{x}\, dx = \frac{\pi a}{2}$, where $f(s)$ is Fourier transform of $F(x)$.

2. Use Parseval's identities to prove that

$$\text{(i)} \quad \int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sin a\,t}{t(a^2+t^2)}\, dt = \frac{\pi}{2} \times \frac{1-e^{-a^2}}{a^2}$$

Fourier transforms of the derivatives of a function. Derivative Theorems.

Theorem 1. If $F\{F(x)\} = f(s), then F\{F'(x)\} = -isf(s)$.

Or, If $F(x)$ has the Fourier transform $f(s)$, then the Fourier transform of $F'(x)$, the derivative of $F(x)$ is $-is\ \ F\{F(x)\}$.

Theorem 2. $F\{F^{(n)}(x)\} = (-is)^n F\{F(x)\}$, i.e., the Fourier transform of the function d^nF/dx^n is $(-is)^n$ times the Fourier transform of the function $F(x)$ provided that the first $(n-1)$ derivatives of $F(x)$ vanish as $x \to \pm\infty$.

Theorem 3. Let $f_c^n(s)$ and $f_s^n(s)$ be the Fourier cosine and sine transform of the function $f^{(n)}(x)$, i.e., d^nF/dx^n, then

$$\text{(i)} \quad f_c^{2n}(s) = -\sum_{r=0}^{n-1} (-1)^r \alpha_{2n-2r-1} s^{2r} + (-1)^n s^{2n} f_c(s)$$

$$\text{(ii)} \quad f_c^{2n+1}(s) = -\sum_{r=0}^{n} (-1)^r \alpha_{2n-2r} s^{2r} + (-1)^n s^{2n+1} f_s(s)$$

$$\text{(iii)} \quad f_s^{2n}(s) = -\sum_{r=0}^{n} (-1)^r \alpha_{2n-2r} s^{2r-1} + (-1)^{n+1} s^{2n} f_s(s)$$

$$\text{(iv)} \quad f_s^{2n+1}(s) = -\sum_{r=0}^{n} (-1)^r \alpha_{2n-2r+1} s^{2r-1} + (-1)^{n+1} s^{2n+1} f_c(s),$$

provided that the first $(n-1)$ derivatives of $f(x)$ vanish as $x \to \infty$ and $F^{(n-1)}(x) \to \alpha_{n-1}$ etc. as $x \to 0$.

Theorem 4. To show that $F\left\{\int_a^x F(x)\,dx\right\} = \frac{f(s)}{(-is)}$, where $f(s)$ is Fourier transform of $F(x)$.

Multiple Fourier transforms. The theory of Fourier transforms of functions of a single variable can be extended to functions of several variables. Let $F(x,y)$ be a function of two variables. To begin with, we treat it temporarily as a function of x, then

$$\overline{F}_x(p,y) = \int_{-\infty}^{\infty} F(x,y)e^{ipx}\,dx,$$

where $\overline{F}_x(p,y)$ stands for Fourier complex transform of $F(x,y)$ with respect to x. Now, regarding $\overline{F}_x(p,y)$ as a function of y, its complex Fourier transform $\overline{F}(p,q)$ is given by

$$\overline{F}(p,q) = \int_{-\infty}^{\infty} \overline{F}_x(p,y)e^{iqy}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)e^{i(px+qy)}\,dx\,dy$$

and the corresponding inversion formula is given by

$$F(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F}(p,q)e^{-i(px+qy)}\,dp\,dq.$$

Note. When $F(x,y)$ is a function of two variables we shall use new variables p and q while finding Fourier transform w.r.t. 'x' and 'y' respectively in place of single variable s as before.

Module 4

Applications of Laplace and Fourier transform to PDEs

Topics & Contents:

1. Apllications to finite difference schemes to solve heat,wave and laplace equations.

Some Useful results for direct applications: We first discuss the use of the (infinite) sine and cosine transform and later on examples will be presented in which the complex Fourier transform has to be used.

The (infinite) sine and cosine transforms can be applied when the range of the variable selected for exclusion (in a given boundary value problem) is 0 to ∞. The choice of sine and cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. In this connection note the following:

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin s\,x\,dx = \left[\frac{\partial u}{\partial x}\sin s\,x\right]_0^\infty - s\int_0^\infty \frac{\partial u}{\partial x}\cos s\,x\,dx, \text{ integrating by parts}$$

$$= -s\int_0^\infty \frac{\partial u}{\partial x}\cos s\,x\,dx, \text{ if } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ which usually happens in physical problems}$$

$$= -s\left\{[u\cos s\,x]_0^\infty + s\int_0^\infty u\sin s\,x\,dx\right\} = s(u)_{x=0} - s^2\bar{u}_s, \text{ assuming that } u \to 0 \text{ as } x \to \infty$$

$$\text{Thus } F_s\{\partial^2 u/\partial x^2\} = s\,u(0,t) - s^2\bar{u}_s(s,t). \qquad \ldots(1)$$

where $u(x,t)$ is a function of two variables x and t and $\bar{u}_s(s,t)$ is the Fourier sine transform of $u(x,t)$ with respect to x.

$$\text{Next, } F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos s\,x\,dx$$

$$= \left[\frac{\partial u}{\partial x}\cos s\,x\right]_0^\infty + s\int_0^\infty \frac{\partial u}{\partial x}\sin s\,x\,dx, \text{ integrating by parts}$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\int_0^\infty \frac{\partial u}{\partial x}\sin s\,x\,dx, \text{ assuming that } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left\{[u\sin s\,x]_0^\infty - s\int_0^\infty u\cos s\,x\,dx\right\} = \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2\int_0^\infty u\,(x,t)\cos s\,x\,dx,$$

assuming that $u \to 0$ as $x \to \infty$

$$\therefore F_c\{\partial^2 u/\partial x^2\} = -(\partial u/\partial x)_{x=0} - s^2\bar{u}_c(s,t), \qquad \ldots(2)$$

where $\bar{u}_c(s,t)$ is the Fourier cosine transform of $u(x,t)$ with respect to x.

From (1) and (2), note carefully that the successful use of a sine transform in removing a term $\partial^2 u/\partial x^2$ from a partial differential equation requires a knowledge of $u(0,t)$ i.e. u when $x = 0$, while the use of a cosine transform for the same purpose requires $u_x(0,t)$ i.e. $\partial u/\partial x$ when $x = 0$.

Remark 1. A term $\partial u/\partial x$ or any partial derivative of odd order cannot be removed with the help of sine or cosine transforms.

Remark 2. When one of the variables in a differential equations ranges from $-\infty$ to ∞ then that variable can be excluded by the help of complex Fourier transforms.

Solved examples based on applications of infinite Fourier transform to boundary value problems.

Example: Solve $\partial u/\partial t = \partial^2 u/\partial x^2, x > 0, t > 0$, subject to conditions

$$u(0,t) = 0, u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}, \; u(x,t) \text{ is bounded}$$

Give a possible physical interpretation of the problem.

Solution: Since $u(0,t)$ i.e. $(u)_{x=0}$ is given, taking the Fourier sine transform of both sides of the given partial differential equations, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \sin s\,x\,dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin s\,x\,dx$$

$$\text{or } \frac{d}{dt}\int_0^\infty u \sin s\,x\,dx = \left[\frac{\partial u}{\partial x}\sin s\,x\right]_0^\infty - s\int_0^\infty \frac{\partial u}{\partial x}\cos s\,x\,dx$$

$$\text{or } \frac{d\bar{u}_s}{dt} = -s\int_0^\infty \frac{\partial u}{\partial x}\cos s\,x\,dx, \text{ if } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

[Here $\bar{u}_s(s,t)$ is the Fourier sine transform of $u(x,t)$]

$$\text{or } \frac{d\bar{u}_s}{dt} = -s\left\{[u\cos s\,x]_0^\infty + s\int_0^\infty u \sin s\,x\,dx\right\} = su(0,t) - s^2\bar{u}_s, \text{ if } u \to 0 \text{ as } x \to \infty$$

$$\text{or } d\bar{u}_s/dt = -s^2\bar{u}_s \quad \text{or} \quad (1/\bar{u}_s)d\bar{u}_s = -s^2 dt, \quad \text{as } u(0,t) = 0$$

whose solution is $\bar{u}_s(s,t) = ce^{-s^2 t}$, c being an arbitrary constant ... (1)

Putting $t = 0$ in (1), we have $c = \bar{u}_s(s,0)$

$$\therefore c = \bar{u}_s(s,0) = \int_0^\infty u\,(x,0)\sin s\,x\,dx = \int_0^1 u\,(x,0)\sin s\,x\,dx + \int_1^\infty u\,(x,0)\sin s\,x\,dx$$

$$= \int_0^1 \sin s\,x\,dx, \text{ using the given value of } u(x,0).$$

$$\therefore c = [(-1/s)\cos s\,x]_0^1 = (1-\cos s)/s$$

$$\therefore (1) \text{ becomes } \bar{u}_s(s,t) = \{(1-\cos s)/s\}e^{-s^2t}$$

Now, taking the inverse Fourier sine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos s}{s} e^{-s^2t} \sin s\,x\,ds, \text{ which is the required solution.}$$

Physical interpretation of the given boundary value problem: $u(x,t)$ represents the temperature at any point x at any time t in a solid $x > 0$. The condition $u(0,t) = 0$ expresses the fact that end $x = 0$ is kept at temperature zero, while the value of $u(x,0)$ expresses the initial temperature as a function of x. Finally $u(x,t)$ is bounded shows that physically temperature of given solid can never be infinite.

Example: Solve $\partial U/\partial t = 2(\partial^2 U/\partial x^2)$, if $U(0,t) = 0, U(x,0) = e^{-x}, x > 0, U(x,t)$ in bounded where $x > 0, t > 0$.

Solution: Given

$$\partial U/\partial t = 2(\partial^2 U/\partial x^2),\ x > 0, t > 0, \qquad \ldots(1)$$

with the boundary conditions: $U(0,t) = 0, \quad U(x,t)$ is bounded $\qquad \ldots(2)$

and initial condition: $U(x,0) = e^{-x}, x > 0 \qquad \ldots(3)$

Since $U(0,t)$ is given, we take the Fourier sine transform of both sides of (1) and obtain.

$$\int_0^\infty \frac{\partial U}{\partial t}\sin s\,x\,dx = 2\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin s\,x\,dx$$

$$\text{or } \frac{d}{dt}\int_0^\infty U(x,t)\sin s\,x\,dx$$

$$= 2\left\{\left[\frac{\partial U}{\partial x}\sin s\,x\right]_0^\infty - \int_0^\infty \frac{\partial U}{\partial x}(s\cos s\,x)\,dx\right\}, \text{ integrating by parts}$$

$$\text{or } \frac{d\bar{U}_s}{dt} = -2s\int_0^\infty \frac{\partial U}{\partial x}\cos s\,x\,dx, \text{ if } \frac{\partial U}{\partial x} \to 0 \text{ as } x \to \infty, \text{ assuming } \bar{U}_s(s,t)$$

$$= \int_0^\infty U(x,t)\sin s\,x\,dx$$

$$\text{or } \frac{d\bar{U}_s}{dt} = -2s\left\{[U(x,t)\cos s\,x]_0^\infty - \int_0^\infty U(x,t)\cdot(-s\sin s\,x)\,dx\right\}$$

or $\frac{d\bar{U}_s}{dt} = -2s\left\{0 - U(0,t) + s\int_0^\infty U(x,t)\sin s\,x\,dx\right\}$, since given that $U(x,t) \to 0$ as $x \to \infty$

or $d\bar{U}_s/dt = 2sU(0,t) - 2s^2\bar{U}_s = -2s^2\bar{U}_s$, as from (2), $U(0,t) = 0$

or $(1/\bar{U}_s)d\bar{U}_s = -2s^2dt.$

Integrating, $\log\bar{U}_s - \log c = -2s^2t$ or $\bar{U}_s(s,t) = ce^{-2s^2t}$ ... (4)

Taking the Fourier sine transform of both sides of (3), we get

$$\int_0^\infty U(x,0)\sin s\,x\,dx = \int_0^\infty e^{-x}\sin s\,x\,dx$$

$$\text{or } \bar{U}_s(s,0) = \left[\frac{e^{-x}}{1+s^2}(-\sin s\,x - s\cos s\,x)\right]_0^\infty = \frac{s}{1+s^2} \quad \ldots(5)$$

Putting $t = 0$ in (4) and using (5), we get $c = s/(1+s^2)$.

$$\therefore \text{From (4), } \bar{U}_s(s,t) = \{s/(1+s^2)\}e^{-2s^2t}.$$

Now, taking the inverse Fourier sine transform, we get

$$U(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}e^{-2s^2t}\sin s\,x\,ds.$$

Example: Using the Fourier sine transform, solve the partial differential equation $\partial V/\partial t = k(\partial^2V/\partial x^2)$ for $x > 0, t > 0$, under the boundary conditions $V = V_0$ when $x = 0, t > 0$ and the initial condition $V = 0$, when $t = 0, x > 0$.

Solution: given

$$\partial V/\partial t = k(\partial^2V/\partial x^2),\ x > 0, t > 0, \quad \ldots(1)$$

with the boundary condition: $V(0,t) = V_0$, ... (2)

and initial condition : $V(x,0) = 0$. ... (3)

Taking the Fourier sine transform of both sides of (1), we get

$$\int_0^\infty \frac{\partial V}{\partial t}\sin s\,x\,dx = k\int_0^\infty \frac{\partial^2V}{\partial x^2}\sin s\,x\,dx$$

$$\text{or } \frac{d}{dt}\int_0^\infty V(x,t)\sin s\,x\,dx = k\left[\left[\frac{\partial V}{\partial x}\sin s\,x\right]_0^\infty - \int_0^\infty \frac{\partial V}{\partial x}s\cos s\,x\,dx\right], \text{ integrating by parts}$$

$$\text{or } \frac{d\bar{V}_s}{dt} = -ks\int_0^\infty \frac{\partial V}{\partial x}\cos s\,x\,dx, \text{ if } \frac{\partial V}{\partial x} \to 0, \text{ as } x \to \infty, \text{ assuming } \bar{V}_s(s,t)$$

$$= \int_0^\infty V\,(x,t)\sin s\,x\,dx$$

$$\text{or } \frac{d\bar{V}_s}{dt} = -ks\left\{[V(x,t)\cos s\,x]_0^\infty - \int_0^\infty V\,(x,t)\cdot(-s\sin s\,x)\,dx\right\}$$

$$\text{or } \frac{d\bar{V}_s}{dt} = -ks\left\{0 - V(0,t) + s\int_0^\infty V\,(x,t)(-s\sin s\,x)\,dx\right\} \text{ assuming that } V(x,t) \to 0 \text{ as } x \to \infty$$

$$\text{or } d\bar{V}_s/dt = ksV_0 - ks^2\bar{V}_s \quad \text{or} \quad d\bar{V}_s/dt + ks^2\bar{V}_s = ksV_0, \text{ by (2)}$$

which is linear differential equation whose I.F. $= e^{\int ks^2 dt} = e^{ks^2 t}$

$$\text{and solution is } e^{ks^2 t}\bar{V}_s(s,t) = \int (ksV_0)\,e^{ks^2 t}\,dt = \frac{ksV_0}{ks^2}e^{ks^2 t} + c$$

$$\text{or } \bar{V}_s(s,t) = V_0/s + ce^{-ks^2 t}. \qquad \ldots(4)$$

Taking the Fourier sine transform of condition (3), we have

$$\int_0^\infty V\,(x,0)\sin s\,x\,dx = 0 \quad \text{or} \quad \bar{V}_s(s,0) = 0. \qquad \ldots(5)$$

Putting $t = 0$ in (4) and using (5), $0 = V_0/s + c$ so that $c = -V_0/s$.

$$\therefore \text{From (4), } \bar{V}_s(s,t) = (V_0/s)\left(1 - e^{-ks^2 t}\right)$$

Now, taking the inverse Fourier sine transform, we get

$$V(x,t) = \frac{2}{\pi}\int_0^\infty \frac{V_0}{s}\left(1 - e^{-ks^2 t}\right)\sin s\,x\,ds = \frac{2V_0}{\pi}\left[\int_0^\infty \frac{\sin s\,x}{s}\,ds - \int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin s\,x\,ds\right]$$

$$= \frac{2V_0}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin s\,x\,ds\right], \quad \text{as} \quad \int_0^\infty \frac{\sin s\,x}{s}\,dx = \frac{\pi}{2}$$

$$\therefore V(x,t) = V_0\left[1 - \frac{2}{\pi}\int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin s\,x\,ds\right] \qquad \ldots(7)$$

Remark. The result (7) can also be expressed in terms of error function or complementary error function as follows.

We know that

$$\int_0^\infty e^{-\alpha^2 x^2}\cos 2\,\beta x\,dx = \frac{\sqrt{\pi}}{2\alpha}e^{-\beta^2/\alpha^2} \qquad \ldots(8)$$

Integrating both sides of (8) w.r.t. β from 0 to β, we get

$$\int_0^{\beta}\left[\int_0^{\infty} e^{-\alpha^2x^2}\cos 2\beta x\, dx\right]d\beta = \frac{\sqrt{\pi}}{2\alpha}\int_0^{\beta} e^{-\beta^2/\alpha^2}\, d\beta$$

$$\text{or } \int_0^{\infty} e^{-\alpha^2x^2}\left[\int_0^{\beta}\cos 2\beta x\, d\beta\right]dx = \frac{\sqrt{\pi}}{2\alpha}\int_0^{\beta/\alpha} e^{-u^2}\,\alpha\, du$$

[on R.H.S putting $\beta/\alpha = u$ so that $d\beta = \alpha\, du$ and changing the order of integration on L.H.S.]

$$\text{or } \int_0^{\infty} e^{-\alpha^2x^2}\left[\frac{\sin 2\beta x}{2x}\right]_0^{\beta} dx = \frac{\sqrt{\pi}}{2}\int_0^{\beta/\alpha} e^{-u^2}\, du \quad \text{or} \quad \int_0^{\infty}\frac{e^{-\alpha^2x^2}\sin 2\beta x}{x}dx$$
$$= \frac{\pi}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{\beta/\alpha} e^{-u^2}\, du$$

$$\text{or } \int_0^{\infty}\frac{e^{-\alpha^2x^2}\sin 2\beta x}{x}dx = \frac{\pi}{2}\text{erf}(\beta/\alpha), \quad \text{as} \quad \text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^{t} e^{-x^2}\, dx$$

$$\text{or } \int_0^{\infty}\frac{e^{-\alpha^2s^2}\sin 2\beta s}{s}ds = \frac{\pi}{2}\text{erf}\left(\frac{\beta}{\alpha}\right) \cdot \text{ or } \quad \int_0^{\infty}\frac{e^{-kts^2}\sin s\, x}{s}ds = \frac{\pi}{2}\text{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

taking $\alpha = \sqrt{kt}$ and $\beta = x/2$

Hence (7) becomes

$$V(x,t) = V_0\left[1 - \text{erf}\left(x/2\sqrt{kt}\right)\right] \qquad \ldots(9)$$

$$\text{or } V(x,t) = V_0\,\text{erfc}\left(x/2\sqrt{kt}\right) \qquad \ldots(10)$$

(9) and (10) are modified forms of solution (7) in terms of error function and complementary error function respectively.

Example: Show that the solution of Laplace equation for V inside the semi-infinite strip $x > 0$, $0 < y < b$, such that $V = f(x)$, when $y = 0, 0 < x < \infty$; $V = 0$, when $y = b, 0 < x < \infty$; $V = 0$, when $x = 0, 0 < y < b$, is given by

$$V = \frac{2}{\pi}\int_0^{\infty} f(u)du\int_0^{\infty}\frac{\sinh(b-y)\,s}{\sinh s\, b}\sin x\, s\sin u\, s\, ds.$$

Solution: Here we are to solve two-dimensional Laplace equation

$$\partial^2V/\partial x^2 + \partial^2V/\partial y^2 = 0, \quad 0 < x < \infty, \quad 0 < y < b \qquad \ldots(1)$$

with the boundary conditions: $V(x,0) = f(x), 0 < x < \infty,$ $\ldots(2)$

$$V(x,b) = 0, 0 < x < \infty \qquad \ldots(3)$$

and $V(0,y) = 0, 0 < y < b.$ $\ldots(4)$

[Here we have variable y in place of t. We shall take Fourier sine transform w.r.t. x and keep y unchanged as we used to do with variable t]

Taking the Fourier sine transform of (1), we get

$$\int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin s\,x\,dx + \int_0^\infty \frac{\partial^2 V}{\partial y^2} \sin s\,x\,dx = 0$$

$$\left[\frac{\partial V}{\partial x} \sin s\,x\right]_0^\infty - \int_0^\infty \frac{\partial V}{\partial x} s \cos s\,x\,dx + \frac{d^2}{dy^2}\int_0^\infty V(x,y) \sin s\,x\,dx = 0, \text{ integrating by parts}$$

$$\text{or } -s\int_0^\infty \frac{\partial V}{\partial x} \cos s\,x\,dx + \frac{d^2 \bar{V}_s}{dy^2} = 0, \text{ if } \frac{\partial V}{\partial x} \to 0 \text{ as } x \to \infty, \text{ assuming } \bar{V}_s(s,y) = \int_0^\infty V(x,y) \sin s\,x\,dx$$

$$\text{or } -s\left\{[V(x,y)\cos s\,x]_0^\infty - \int_0^\infty V(x,y)(-s \sin s\,x)\,dx\right\} + \frac{d^2 \bar{V}_s}{dy^2} = 0, \text{ integrating by parts}$$

$$-s\{-V(0,y) + s\bar{V}_s\} + (d^2\bar{V}_s/dy^2) = 0, \text{ if } V \to 0 \text{ as } x \to \infty$$

$$-s^2 + (d^2\bar{V}_s/dy^2) = 0, \text{ using (4)}$$

$$(D^2 - s^2)\bar{V}_s = 0,\ D \equiv d/dy, \text{ whose general solution is}$$

$$\bar{V}_s(s,y) = A\cosh s\,y + B \sinh s\,y, \text{ where } A \text{ and } B \text{ are arbitrary constants} \qquad \ldots(5)$$

Taking the Fourier sine transform of (2) and (3), we get

$$\int_0^\infty V(x,0) \sin s\,x\,dx = \int_0^\infty f(x) \sin s\,x\,dx \quad \text{and} \quad \int_0^\infty V(x,b) \sin s\,x\,dx = 0$$

$$\bar{V}_s(s,0) = \int_0^\infty f(u) \sin s\,u\,du \qquad \ldots(6)$$

$$\bar{V}_s(s,b) = 0 \qquad \ldots(7)$$

Putting $y = 0$ in (5) and using (6), we get $\int_0^\infty f(u) \sin s\,u\,du = A.$ $\ldots(8)$

Putting $y = b$ in (5) and using (7) and (8), we get

$$0 = \cosh s\,b \int_0^\infty f(u) \sin s\,u\,du + B \sinh s\,b \quad \Rightarrow \quad B = -\frac{\cosh s\,b}{\sinh s\,b}\int_0^\infty f(u) \sin s\,u\,du \qquad \ldots(9)$$

Putting values of A and B in (5), we get

$$\bar{V}_s(s,y) = \cosh s\,y \int_0^\infty f(u) \sin s\,u\,du - \frac{\sinh s\,y \cosh s\,b}{\sinh s\,b}\int_0^\infty f(u) \sin s\,u\,du$$

$$\therefore \bar{V}_s(s,y) = \int_0^\infty f(u) \sin s\,u \left[\cosh s\,y - \frac{\sinh s\,y \cosh s\,b}{\sinh s\,b}\right] du$$

$$= \int_0^\infty f(u) \sin s\,u \frac{\sinh(b-y)\,s}{\sinh s\,b}\, du.$$

Now, taking the inverse Fourier sine transform, we get

$$V(x,y) = \frac{2}{\pi}\int_0^\infty \bar{V}_s(x,y) \sin s\,x\, ds = \frac{2}{\pi}\int_0^\infty \sin s\,x \left[\int_0^\infty f(u) \frac{\sin s\,u \sinh(b-y)\,s}{\sinh s\,b}\, du\right] ds$$

$V(x,y)$

$= \frac{2}{\pi}\int_0^\infty f(u)\, du \int_0^\infty \frac{\sinh(b-y)\,s}{\sinh s\,b} \sin s\,u \sin s\,x\, ds$, on changing the order of integration

Example: The temperature U in the semi-infinite rod $0 \le x < \infty$ is determined by the equation $\partial U/\partial t = k(\partial^2 U/\partial x^2)$ subject to the conditions.

(i) $U = 0$ when $t = 0, x \ge 0$

(ii) $\partial U/\partial x = -\mu$ (a constant), when $x = 0$ and $t > 0$.

Making use of cosine transform, show that $U(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos sx}{s^2}\left(1 - e^{-ks^2 t}\right) ds$

Solution: given

$$\partial U/\partial t = k(\partial^2 U/\partial x^2) \qquad \ldots (1)$$

$$(\partial U/\partial x)_{x=0} = -\mu \qquad \ldots (2)$$

with the boundary condition: $U(x,0) = 0, x \ge 0.$ $\ldots (3)$

and initial conditions: Taking the Fourier cosine transforms of both sides of (1), we get

$$\int_0^\infty \frac{\partial U}{\partial t} \cos s\,x\, dx = k\int_0^\infty \frac{\partial^2 U}{\partial x^2} \cos s\,x\, dx$$

or $\frac{d}{dt}\int_0^\infty U(x,t) \cos s\,x\, dx$

$$= k\left\{\left[\frac{\partial U}{\partial x}\cos s\,x\right]_0^\infty - \int_0^\infty \frac{\partial U}{\partial x}(-s \sin s\,x)\, dx\right\}, \text{ integrating by parts}$$

or $\frac{d\bar{U}_c}{dt} = -k\left(\frac{\partial U}{\partial x}\right)_{x=0} + ks\int_0^\infty \frac{\partial U}{\partial x} \sin s\,x\, dx$, if $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$, assuming $\bar{U}_c$

$$= \int_0^\infty U(x,t) \cos s\,x\, dx$$

or $\frac{d\bar{U}_c}{dt} = k\mu + ks\left\{[U(x,t) \sin s\,x]_0^\infty - \int_0^\infty U(x,t) s \cos s\,x\, dx\right\}$, by (2)

or $d\bar{U}_c/dt = k\mu + ks(0 - s\bar{U}_c)$, if $U \to 0$ as $x \to \infty$

or $(d\bar{U}_c/dt) + ks^2\bar{U}_c = k\mu$, which is a linear equation ... (4)

I.F. of (4) $= e^{\int ks^2 dt} = e^{ks^2 t}$ and so general solution of (4) is

$$e^{ks^2 t}\bar{U}_c(s,t) = \int (k\mu)\, e^{ks^2 t} dt + c = k\mu \frac{e^{ks^2 t}}{ks^2} + c.$$

$$\therefore \bar{U}_c(s,t) = \mu/s^2 + ce^{-ks^2 t} \qquad \ldots (5)$$

Taking the Fourier cosine transform of (3), we get

$$\int_0^\infty U(x,0) \cos s\, x\, dx = 0 \quad \text{or} \quad \bar{U}_c(s,0) = 0. \qquad \ldots (6)$$

Putting $t = 0$ in (5) and using (6), $0 = \mu/s^2 + c$ so that $c = -\mu/s^2$.

$$\therefore \text{From (5), } \bar{U}_c(s,t) = (\mu/s^2) \times \left(1 - e^{-ks^2 t}\right).$$

$\therefore$ Taking the inverse Fourier cosine transform, we get

$$U(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}\left(1 - e^{-ks^2 t}\right) \cos s\, x\, ds = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos s\, x}{s^2}\left(1 - e^{-ks^2 t}\right) ds$$

Example: Solve $\partial u/\partial t = \partial^2 u/\partial x^2, x > 0, t > 0$, subject to conditions $u_x(0,t) = 0, u(x,t)$ is bounded and $u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Solution: Since $u_x(0,t)$ is given, taking the Fourier cosine transform of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \cos s\, x\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos s\, x\, dx$$

$$\text{or } \frac{d}{dt}\int_0^\infty u \cos s\, x\, dx = \left[\frac{\partial u}{\partial x} \cos s\, x\right]_0^\infty + s\int_0^\infty \frac{\partial u}{\partial x} \sin s\, x\, dx, \text{ integrating by parts}$$

$$\text{or } \frac{d\bar{u}_c}{dt} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left[[u \sin s\, x]_0^\infty - s\int_0^\infty u \cos s\, x\, dx\right], \qquad \ldots (1)$$

where we have assumed that $\partial u/\partial x \to 0$ as $x \to \infty$ and $\bar{u}_c(s,t)$ is the Fourier cosine transform of $u(x,t)$.

$$\therefore (1) \Rightarrow d\bar{u}_c/dt = -s^2\bar{u}_c \quad \text{or} \quad (1/\bar{u}_c)d\bar{u}_c = -s^2 dt.$$

[$\because$ given $u_x(0,t) = 0$ and let $u \to 0$ as $x \to \infty$]

Integrating, $\log \bar{u}_c - \log A = -s^2 t$ or $\bar{u}_c(s,t) = Ae^{-s^2 t}$. ... (2)

Putting $t = 0$ in (2), we get $A = \bar{u}_c(s, 0)$. Hence, using definition, we have

$$A = \bar{u}_c(s,0) = \int_0^\infty u\,(x,0)\cos s\,x\,dx = \int_0^1 u\,(x,0)\cos s\,x\,dx + \int_1^\infty u\,(x,0)\cos s\,x\,dx$$
$$= \int_0^1 x\cos s\,x\,dx$$

[using the given value of $u(x, 0)$]

$$= \left[x\frac{\sin s\,x}{s}\right]_0^1 - \int_0^1 1\times\frac{\sin s\,x}{s}\,dx = \frac{\sin s}{s} + \left[\frac{\cos s\,x}{s^2}\right]_0^1 = \frac{\sin s}{s} + \frac{\cos s - 1}{s^2}$$

$$\therefore (2) \text{ becomes } \bar{u}_c(s,t) = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t}.$$

Now, taking the inverse Fourier cosine transforms, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t}\cos s\,x\,dx$$

$$\text{or } s\left\{[U(x,y)\sin s\,x]_0^\infty - \int_0^\infty U\,(x,y)s\cos s\,x\,dx\right\} + (d^2\bar{U}_c/dy^2)$$
$$= 0, \text{ integrating by parts again,}$$

$$0. -s^2\bar{U}_c + (d^2\bar{U}_c/dy^2) = 0, \text{ if } U(x,y) \to 0 \text{ as } x \to \infty$$

$$\text{or } (D^2 - s^2)\bar{U}_c = 0, \text{ where } D \equiv d/dy.$$

whose general solution is $\bar{U}_c(s,y) = C_1 e^{sy} +$
$C_2 e^{-sy}, C_1, C_2$ being arbitrary constants ... (4)

Since $\bar{U}_c(s, y)$ is finite, we must take $C_1 = 0$ in (4), otherwise $\bar{U}_c(s, y)$ would become infinite as $y \to \infty$.

$$\therefore (4) \text{ reduces to } \bar{U}_c(s,y) = C_2 e^{-sy} \qquad \ldots (5)$$

$$\text{Again, } \int_0^\infty U\,(x,0)\cos s\,x\,dx = \int_0^a U\,(x,0)\cos s\,x\,dx + \int_a^\infty U\,(x,0)\cos s\,x\,dx$$

$$\text{or } \bar{U}_c(s,0) = \int_0^a \cos s\,x\,dx + 0, \text{ by (2) and (3)}$$

$$\text{or } \bar{U}_c(s,0) = [(1/s)\times\sin s\,x]_0^a = (1/s)\times\sin s\,a. \qquad \ldots (6)$$

Putting $y = 0$ in (5) and using (6), we get $C_2 = (\sin s\,a)/s$.

$$\therefore \text{From (5), } \bar{U}_c(s,y) = \{(1/s)\times\sin s\,a\}e^{-sy}.$$

$$\therefore \text{Taking the inverse Fourier cosine transform, we get}$$

$$U(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sin s\,a}{s}\,e^{-sy}\cos s\,x\,ds = \frac{1}{\pi}\int_0^\infty \frac{e^{-sy}}{s}[\sin(a+x)\,s + \sin(a-x)\,s]ds$$

$$= \frac{1}{\pi}\left[\int_0^\infty e^{-sy}\, s^{-1} \sin(a+x)\, s\, ds + \int_0^\infty e^{-sy}\, s^{-1} \sin(a-x)\, s\, ds\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right], \text{ as } \int_0^\infty e^{-sx}\, x^{-1} \sin r\, x\, dx = \tan^{-1}\frac{r}{s}$$

Example: If the flow of heat is linear so that the variation of θ (temperature) with z and y may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation $\partial\theta/\partial t = k(\partial^2\theta/\partial x^2)$ where $-\infty < x < \infty$ and $\theta = f(x)$ when $t = 0$, $f(x)$ being a given function of x.

Solution: given $\partial\theta/\partial t = k(\partial^2\theta/\partial x^2), -\infty < x < \infty,$ ... (1)

with the initial condition: $\theta(x,0) = f(x)$. ... (2)

Taking the complex Fourier transform of (1), we get

$$\int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t} e^{isx} dx = k\int_{-\infty}^{\infty} \frac{\partial^2\theta}{\partial x^2} e^{isx} dx$$

or $\frac{d}{dt}\int_{-\infty}^{\infty} \theta\,(x,t) e^{isx} dx = k(-is)^2 F\{\theta(x,t)\}$, using theorem 2, Art. 1.22

or $\frac{d\bar{\theta}}{dt} = -ks^2\bar{\theta}$, where $\bar{\theta}(s,t) = F\{\theta(x,t)\} = \int_{-\infty}^{\infty} \theta\,(x,t) e^{isx} dx$

whose general solution is $\bar{\theta}(s,t) = Ae^{-ks^2t}$, A being an arbitrary constant ... (3)

Taking the complex Fourier transform of both sides of (2), we get

$$\int_{-\infty}^{\infty} \theta\,(x,0) e^{isx} dx = \int_{-\infty}^{\infty} f\,(x) e^{isx} dx \quad \text{or} \quad \bar{\theta}(s,0) = \bar{f}(s) \qquad \ldots (4)$$

Putting $t = 0$ in (3) and using (4), we get $\bar{f}(s) = A$.

$$\therefore \text{ From (3), } \bar{\theta}(s,t) = \bar{f}(s) e^{-ks^2t}$$

Taking the inverse Fourier transform, we get

$$\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}\,(s) e^{-ks^2t} e^{-isx} ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}\,(s) e^{-ks^2t - isx} ds,$$

where $\bar{f}(s) = \int_{-\infty}^{\infty} f\,(x) e^{isx} dx$.

Example: If the function $U(x,y)$ is determined by equation $\partial U/\partial x = \partial^2 U/\partial y^2$ for $x \geq 0, -\infty < y < \infty$ and $U = f(y)$ when $x = 0$, show that

$$U(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s) e^{-s^2x - isy} ds, \text{ where } \bar{f}(s) \text{ is the Fourier transform of } f(y).$$

Example: Solve $\partial^4 V/\partial x^4 + \partial^2 V/\partial y^2 = 0, -\infty < x < \infty, y \geq 0$ satisfying the conditions.

(i) V and its partial derivatives tend to zero as $x \to \pm\infty$ and (ii) $V = f(x), \partial V/\partial y = 0$ on $y = 0$.

Solution: given $\partial^4 V/\partial x^4 + \partial^2 V/\partial y^2 = 0$ ... (1)

with the boundary conditions:

$$V \to 0, \quad \partial V/\partial x \to 0, \quad \partial^2 V/\partial x^2 \to 0, \quad \partial^3 V/\partial x^3 \to 0, \quad \text{as } x \to \pm\infty, \qquad \ldots (2)$$

$$V(x, 0) = f(x) \qquad \ldots (3)$$

and $(\partial V/\partial y)_{y=0} = 0$ ... (4)

Taking the complex Fourier transform of (1), we get

$$\int_0^\infty \frac{\partial^4 V}{\partial x^4} e^{isx} dx + \int_0^\infty \frac{\partial^2 V}{\partial y^2} e^{isx} dx = 0$$

Or $(-is)^4 F\{V(x, y)\} + \frac{d^2}{dy^2} \int_0^\infty V(x, y) e^{isx} dx = 0$, by theorem 2 of Art. 6.14.

Or $s^4 \bar{V} + \frac{d^2 \bar{V}}{dy^2} = 0$, where $F\{V(x, y)\} = \bar{V}(s, y) = \int_0^\infty V(x, y) e^{isx} dx$.

Or $(D^2 + s^4)\bar{V} = 0$, where $D \equiv d/dy$,

whose general solution is given by $\bar{V}(s, y) = C_1 \cos s^2 y + C_2 \sin s^2 y$ ... (5)

From (5), $d\bar{V}/dy = -C_1 s^2 \sin s^2 y + C_2 s^2 \cos s^2 y$. ... (6)

Taking complex Fourier transform of (4), we get

$$\int_0^\infty \left(\frac{\partial V}{\partial y}\right)_{y=0} e^{isx} dx = 0 \quad \text{or} \quad \left(\frac{d\overline{V}}{dy}\right)_{y=0} = 0$$

Putting $y = 0$ in (6) and using (7), we get $0 = C_2 s^2$ so that $C_2 = 0$.

∴ From (5),

$$\overline{V}(s, y) = C_1 \cos s^2 y$$

Taking complex Fourier transform of (3), we get

$$\int_0^\infty V(x, t) e^{isx} dx = \int_0^\infty f(x) e^{isx} dx \quad \text{or} \quad \overline{V}(s, 0) = \overline{f}(s)$$

where

$$\overline{f}(s) = \int_0^\infty f(x) e^{isx} dx$$

Putting $y = 0$ in (8) and using (9), we get $\overline{f}(s) = C_1$

∴ From (8), we have

$$\overline{V}(s, y) = \overline{f}(s) \cos s^2 y$$

Taking the inverse Fourier transform, we get

$$V(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f}\,(s) \cos s^2 y\, e^{-isx} ds, \quad \text{where } \overline{f}(s) \text{ is given by (10).}$$

Example: Use the complex form of the Fourier transform to show that

$$\mathrm{V} = \frac{1}{2\sqrt{\pi \mathrm{t}}} \int_{-\infty}^{\infty} \mathrm{f}(\mathrm{u}) \exp\left\{-\frac{(\mathrm{x}-\mathrm{u})^2}{4\mathrm{t}}\right\} \mathrm{du}$$

is the solution of the boundary value problem

$$\frac{\partial \mathrm{V}}{\partial \mathrm{t}} = \frac{\partial^2 \mathrm{V}}{\partial \mathrm{x}^2}, \quad -\infty < \mathrm{x} < \infty, \quad \mathrm{t} > 0; \quad \mathrm{V} = \mathrm{f}(\mathrm{x}); \quad \text{when } \mathrm{t} = 0.$$

Solution: Taking the complex Fourier transform of both sides of the given partial differential equation, we have

$$\int_{-\infty}^{\infty} \frac{\partial V}{\partial t} e^{isx} dx = \int_{-\infty}^{\infty} \frac{\partial^2 V}{\partial x^2} e^{isx} dx$$

or

$$\frac{d}{dt} \int_{-\infty}^{\infty} V\, e^{isx} dx = (-is)^2 F\{V(x, t)\}, \text{ by theorem 2 of Art. 1.22}$$

or

$$\frac{d\overline{V}}{dt} = -s^2 \overline{V}, \quad \text{where} \quad \overline{V}(s, t) = \int_{-\infty}^{\infty} V\,(x, t) e^{isx} dx$$

or

$$\left(1/\overline{V}\right) d\overline{V} = -s^2 dt \quad \text{so that} \quad \overline{V}(s, t) = A e^{-s^2 t}$$

Putting $t = 0$ in (1), we get $A = \overline{V}(s, 0)$

∴

$$A = \overline{V}(s, 0) = \int_{-\infty}^{\infty} V\,(x, 0) e^{isx} dx = \int_{-\infty}^{\infty} f\,(x) e^{isx} dx = \int_{-\infty}^{\infty} f\,(u) e^{isu} du$$

$$[\because V = f(x) \text{ when } t = 0 \Rightarrow V(x, 0) = f(x)]$$

∴ From (2)

$$\overline{V}(s,t) = e^{-s^2 t}\int_{-\infty}^{\infty} f(u)e^{isu}du$$

Taking the inverse Fourier transform, we have

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-s^2 t}\int_{-\infty}^{\infty} f(u)e^{isu}du\right]e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-s^2 t - is(x-u)}\,ds\right]du$$

[changing the order of integration]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s^2 + is(x-u)/t\right]}\,ds\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\{[s+i(x-u)/2t]^2 + (x-u)^2/4t^2\}}\,ds\right]du$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/4t}\left[\int_{-\infty}^{\infty} e^{-t\{s+i(x-u)/2t\}^2}\,ds\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}} f(u)e^{-(x-u)^2/4t}\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right]du$$

$$\left[\text{Putting } \sqrt{t}\{s + i(x-u)/2t\} = y \quad \text{so that } ds = dy/\sqrt{t}\right]$$

$$\frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/4t}\sqrt{\pi}\,du, \quad \text{as } \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\exp\left\{-\frac{(x-u)^2}{4t}\right\}du.$$

Example: Use the method of Fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that solution can also be put in the form $y(x,t) = [f(x+ct) + f(x-ct)]/2$.

Solution: Here we are to solve one-dimensional wave equation

$$\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2), \quad -\infty < x < \infty, \quad t > 0$$

subject to the following initial conditions:

$$y(x,0) = \text{Initial displacement} = f(x)$$

and

$$y_t(x,0) = \text{Initial velocity} = 0$$

Taking the complex Fourier transform of (1), we get

$$\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{isx}dx = c^2\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{isx}dx$$

Or

$$\frac{d^2}{dt^2}\int_{-\infty}^{\infty} y\, e^{isx}dx = (-is)^2c^2F\{y(x,t)\}, \text{ by theorem 2 of Art. 1.22}$$

or

$$\frac{d^2\overline{y}}{dt^2} + c^2s^2\overline{y} = 0, \quad \text{where} \quad F\{y(x,t)\} = \overline{y}(s,t) = \int_{-\infty}^{\infty} y\,(x,t)e^{isx}dx$$

or

$$(D^2 + c^2s^2)\overline{y} = 0, \quad \text{where } D = d/dt. \text{ Its solution is given by}$$

$$\overline{y}(s,t) = A\cos(cst) + B\sin(cst)\text{, where } A \text{ and } B \text{ are arbitrary constants}$$

From (4), $\overline{y}_t(s,t) = \partial\overline{y}/\partial t = -Acs\sin(cst) + Bcs\cos(cst)$.

Taking infinite Fourier transform, (2) and (3), we have

$$\overline{y}(s,0) = \int_{-\infty}^{\infty} f\,(x)e^{isx}dx = \overline{f}(s) \text{ say}$$

and

$$\overline{y}_t(s,0) = \int_{-\infty}^{\infty} 0\,.\,e^{isx}dx = 0.$$

Putting $t = 0$ in (5) and using (7), $0 = Bcs$ so that $B = 0$.

Putting $t = 0$ in (4) and using (6) and $B = 0$, we get $\overline{f}(s) = A$

∴ From (4), $\overline{y}(s,t) = \overline{f}(s)\cos(cst)$.

Taking the inverse Fourier transform, we have

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}\,(s)\cos c\,st\,e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}\,(s)\left(\frac{e^{icst} + e^{-icst}}{2}\right)e^{-isx}ds$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}\,(s)e^{-is(x-ct)}ds + \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}\,(s)e^{-is(x+ct)}ds\right]$$

$$= [f(x+ct) + f(x-ct)]/2, \text{ by inversion formula for Fourier transform.}$$

Example: A thin membrane of great extent is released from rest in position $z = f(x,y)$. Show that displacement at any subsequent time is given by

$$z(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\,(p,q)\cos\{ct(p^2+q^2)^{1/2}\}\,e^{-i(px+qy)}dp\,dq,$$

where $F(p,q)$ is double Fourier transform of $f(x,y)$.

Or

The variation of a function z over the xy-plane and for $t \geq 0$ is determined by the equation

$$\nabla^2 z = (1/c^2) \times (\partial^2 z/\partial t^2), \quad \text{i.e.,} \quad \partial^2 z/\partial t^2 = c^2(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2)$$

where $c^2 = T/\rho$. If when $t = 0$, $z = f(x, y)$ and $\partial z/\partial t = 0$, show that, at any subsequent time,

$$z(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,q)\cos\{ct(p^2+q^2)^{1/2}\}\, e^{-i(px+qy)}dp\,dq,$$

where

$$F(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(px+qy)}dx\,dy \quad \text{and} \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

Solution: We know that the displacement of the membrane satisfies the two-dimensional wave equation

$$\partial^2 z/\partial t^2 = c^2(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2)$$

with given boundary conditions:

$$z(x, y, 0) = f(x, y)$$

and

$$(\partial z/\partial t)_{t=0} = 0$$

Taking double Fourier transform (refer Art. 1.23) of (1) we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 z}{\partial t^2}e^{i(px+qy)}dx\,dy = c^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)e^{i(px+qy)}dx\,dy$$

or

$$\frac{d^2\overline{z}}{dt^2} = c^2(-p^2-q^2)\overline{z} \quad \text{or} \quad [D^2 + c^2(p^2+q^2)]\overline{z} = 0, \quad D \equiv d/dt$$

where

$$\overline{z}(p,q,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(x,y,t)e^{i(px+qy)}dx\,dy$$

is the two-dimensional Fourier transform of $z(x, y, t)$.

Auxiliary equation of (4) is $D^2 + c^2(p^2+q^2) = 0$ giving $D = \pm ic(p^2+q^2)^{1/2}$.

Hence solution of (4) is given by

$$\overline{z}(p,q,t) = C_1\cos\{ct(p^2+q^2)^{1/2}\} + C_2\sin\{ct(p^2+q^2)^{1/2}\},\ C_1, C_2 \text{ being arbitrary constants}$$

Taking the double Fourier transform of conditions (2) and (3), we get

$$\overline{z}(p, q, 0) = F(p, q)$$

and

$$(d\bar{z}/dt)_{t=0} = 0,$$

where

$$F(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(px+qy)}dx\,dy$$

From (6), $d\bar{z}/dt = -C_1 c(p^2+q^2)^{1/2}\sin\{ct(p^2+q^2)^{1/2}\}$

$$+C_2 c(p^2+q^2)^{1/2}\cos\{ct(p^2+q^2)^{1/2}\}.$$

Putting $t = 0$ in (10) and using (8), we get $0 = C_2 c(p^2+q^2)^{1/2}$ so that $C_2 = 0$.

∴ From (6),

$$\bar{z}(p,q,t) = C_1\cos\{ct(p^2+q^2)^{1/2}\}.$$

Putting $t = 0$ in (11) and using (7), we get $C_1 = F(p,q)$.

From (11),

$$\bar{z}(p,q,t) = F(p,q)\cos\{ct(p^2+q^2)^{1/2}\}.$$

By applying inversion theorem for double Fourier transform, we get

$$z(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,q)\cos\{ct(p^2+q^2)^{1/2}\}e^{-i(px+qy)}dp\,dq.$$

Miscellaneous problems :

1. Evaluate $F_s^{-1}\{e^{-ap}/p\}$

2. Find the Fourier cosine and sine transforms of e^{-bx} and show that

$$\text{(i)}\int_0^{\infty}\frac{\cos sx}{s^2+b^2}dx = \frac{\pi}{2b}e^{-bx} \quad \text{(ii)}\int_0^{\infty}\frac{s\sin sx}{s^2+b^2}dx = \frac{\pi}{2}e^{-bx}$$

3. Solve $k(\partial^2 u/\partial x^2) = \partial u/\partial t$, $-\infty < x < \infty, t > 0$ subject to $u(x,t)$ and $u_x(x,t)$ both tend to zero as $|x| \to \infty$ and $u(x,0) = f(x), -\infty < x < \infty$

$$\textbf{Ans.}\quad u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(s)e^{-\{(x-s)^2/4kt\}}ds$$

4. Obtain the temperature distribution in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial temperature is $f(x)$.

Hint. We are to solve $\partial u/\partial t = k(\partial^2 u/\partial x^2)$, $0 < x < \infty, t > 0$ subject to $u(0,t) = 0, t > 0$ and $u(x,0) = f(x), 0 < x < \infty$. Using Fourier sine transform, the required solution is given by

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\bar{f}_s(s)e^{-ks^2t}\sin sx\,ds, \quad \text{where } \bar{f}_s(s) = \int_0^{\infty} f(x)\sin sx\,dx$$

5. Determine the solution of free vibrations of a semi-infinite string described by $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2)$, $0 < x < \infty, t > 0$ subject to $u(x,0) = f(x)$ and $u_t(0,t) = g(x)$.

Ans. $u(x,t) = (1/2) \times \{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$, where $g(s) = \int_0^\infty g(x) \sin s x \, dx$

6 (a) Solve the following boundary value problem in the half-plane $y > 0$, described by $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, $-\infty < x < \infty, y > 0$ subject to $u(x,0) = f(x)$, $-\infty < x < \infty$, u is bounded as $y \to \infty$, u and $\partial u/\partial x$ both vanish as $|x| \to \infty$

(b) Solve the following boundary value problem in the half-plane $x > 0$, described by $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, $-\infty < y < \infty, x > 0$ subject to $u(0,y) = f(y)$, $-\infty < y < \infty$, u is bounded as $x \to \infty$, u and $\partial u/\partial y$ both vanish as $|y| \to \infty$.

Ans. (a) $u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(\xi - x)^2 + y^2}$ (**b**) $u(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{x^2 + (y-\xi)^2}$

Module 5

Numerical Solutions to PDEs

Topics & Contents:

2. Introduction
3. Foeward, Backward and Central difference operator
4. Interpolation
5. Newton's Forward and Backward Interpolation
6. Lagrange's and Newton's divided difference formula for unequal intervals
7. Apllications to finite difference schemes to solve heat,wave and laplace equations.

Introduction: The calculus of finite differences examines how a dependent variable (function) changes in response to finite step changes in an independent variable, contrasting with infinitesimal calculus which deals with continuous variations. This chapter focuses on analyzing functional variations when the independent variable changes by uniform intervals.

Finite Differences: For a function $y = f(x)$ tabulated at equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$, yielding corresponding values $y = y_0, y_1, y_2, \dots, y_n$:

- Forward Differences: The values $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ are denoted as $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$, where Δ represents the forward difference operator.
  - First forward difference: $\Delta y_r = y_{r+1} - y_r$
  - Second forward difference: $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$
  - p-th forward difference: $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$
- **Terminology:**
  - x is the argument.
  - y is the function or entry.
  - y_0 is the leading term.
  - $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$, ... are the leading differences.

- **General Binomial Expansion Form:**

$$\Delta^n y_0 = y_n - \binom{n}{1} y_{n-1} + \binom{n}{2} y_{n-2} - \cdots + (-1)^n y_0$$

Properties of the Operator Δ:

1. Distributive Law: $\Delta[f(x) \pm \phi(x)] = \Delta f(x) \pm \Delta\phi(x)$

2. Constant Rule: $\Delta[c \cdot f(x)] = c \cdot \Delta f(x)$

3. Index Law: $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

Backward Differences:

The backward difference operator is denoted by ∇.

- First Backward Difference:

$$\nabla y_r = y_r - y_{r-1}$$

- Second & Higher-Order Backward Differences:

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$
$$\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1}$$

Central Differences & Mean Operator:

- Central Difference Operator (δ):

$$\delta y_{1/2} = y_1 - y_0$$
$$\delta y_r = y_{r+1/2} - y_{r-1/2}$$

- Mean Operator (μ):

$$\mu y_r = \frac{1}{2}\left(y_{r+1/2} + y_{r-1/2}\right)$$

Forward Difference Table :

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

Backward Difference Table:

X	Y	∇y	∇^2y	∇^3y	∇^4y	∇^5y
x_0	y_0					
$x_0 + h$	y_1	∇y_1				
$x_0 + 2h$	y_2	∇y_2	$\nabla^2 y_2$			
$x_0 + 3h$	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
$x_0 + 4h$	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
$x_0 + 5h$	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

Central Difference Table :

x	Y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

Differences of a Polynomial:

For an n-th degree polynomial $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + k$:

1. The n-th difference is constant:

$$\Delta^n f(x) = a \cdot n! \cdot h^n$$

2. The $n + 1$-th and all higher-order differences are zero:

$$\Delta^{n+1} f(x) = 0$$

Factorial Notation:

A factorial polynomial $[x]^n$ is defined as:

$$[x]^n = x(x-h)(x-2h) \ldots (x-(n-1)h)$$

When the interval of differencing $h = 1$:

$$[x]^n = x(x-1)(x-2) \ldots (x-n+1)$$

Key Rules:

- $\Delta[x]^n = n \cdot h \cdot [x]^{n-1}$
- $\Delta^n[x]^n = n! \cdot h^n$
- $\Delta^{n+1}[x]^n = 0$

Effect of an Error on a Difference Table:

If an error ϵ is present in an entry y_5, it spreads through higher-order differences as follows:

- 1st Difference errors: $+\epsilon, -\epsilon$
- 2nd Difference errors: $+\epsilon, -2\epsilon, +\epsilon$
- 3rd Difference errors: $+\epsilon, -3\epsilon, +3\epsilon, -\epsilon$
- 4th Difference errors: $+\epsilon, -4\epsilon, +6\epsilon, -4\epsilon, +\epsilon$

Key Properties:

- Error magnitudes increase with the order of differencing.
- Errors follow binomial expansion coefficients $(1-\epsilon)^n$.
- The algebraic sum of errors in any single difference column is zero.

Example :

$$\Delta \tan^{-1}(x) = \tan^{-1}(x+h) - \tan^{-1}(x) = \tan^{-1}\left(\frac{h}{1+hx+x^2}\right)$$

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = abcd \cdot (10!)$$

Example :

Expressing $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation yields:

$$y = 2[x]^3 + 3[x]^2 + 2[x] - 10$$

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$$\therefore \text{The other fourth order differences must also be } 16.$$

Thus, i.e.

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

$$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$

$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$

$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and

$$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly, starting with $\Delta^4 y_2 = 16$, we get $\Delta^3 y_3 = 40, \Delta^2 y_4 = 68, \Delta y_5 = 98, y_6 = 129$.

Starting with $\Delta^4 y_3 = 16$, we obtain $\Delta^3 y_4 = 56, \Delta^2 y_5 = 124, \Delta y_6 = 222, y_7 = 351$.

Exercises :

1. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$.
2. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.

3. Express $3x^4 - 4x^3 + 6x^2 + 2x + 1$ as a factorial polynomial and find differences of all orders.
4. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
5. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.
6. Given $\log 100 = 2$, $\log 101 = 2.0043$, $\log 103 = 2.0128$, $\log 104 = 2.0170$, find $\log 102$.
7. Estimate the missing term in the following table:

x:	0	1	2	3	4
$f(x)$:	1	3	9	−	81

8. Find the missing values in the following table:

x:	0	5	10	15	20	25
y:	6	10	−	17	−	31

9. Find the first term of the series whose second and subsequent terms are $8,3,0,-1,0$.
10. Write down the polynomial of lowest degree which satisfies the following set of numbers:

$$0,7,26,63,124,215,342,511.$$

11. The values of a polynomial of degree 5 are tabulated below. If $f(3)$ is known to be in error, find its correct value.

x:	0	1	2	3	4	5	6
$f(x)$:	1	2	33	254	1025	3126	7777

Other difference operators:

We have already introduced the operators Δ, ∇, and δ. Besides these, there are the operators E and μ, which we define below:

(1) Shift operator E is the operation of increasing the argument x by h so that

$$Ef(x) = f(x+h), \quad E^2 f(x) = f(x+2h), \quad E^3 f(x) = f(x+3h) \quad \text{etc.}$$

The inverse operator E^{-1} is defined by $E^{-1}f(x) = f(x-h)$.

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$, where n may be any real number.

(2) Averaging operator μ is defined by the equation:

$$\mu y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}\right)$$

Obs. In the difference calculus, E is regarded as the fundamental operator and $\Delta, \nabla, \delta, \mu$ can be expressed in terms of E.

Relations between the operators: (1) We shall now establish the following identities:

- (i) $\Delta = E - 1$
- (ii) $\nabla = 1 - E^{-1}$

- (iii) $\delta = E^{1/2} - E^{-1/2}$
- (iv) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$
- (v) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$
- (vi) $E = e^{hD}$

Proof:

(i) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation:

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

Obs. These relations imply that the effect of operator E on y_x is the same as that of the operator $1 + \Delta$ on y_x. The operators E and Δ do not have any existence as separate entities.

(ii) $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})y_x$

$$\therefore \nabla = 1 - E^{-1}$$

(iii) $\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h} = E^{1/2}y_x - E^{-1/2}y_x = \left(E^{1/2} - E^{-1/2}\right)y_x$

$$\therefore \delta = E^{1/2} - E^{-1/2}$$

(iv) $\mu y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}\right) = \frac{1}{2}\left(E^{1/2}y_x + E^{-1/2}y_x\right) = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)y_x$

$$\therefore \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

(v) $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$

$$\therefore E\nabla = \Delta$$

$$\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+\frac{1}{2}h} = y_{x+\frac{1}{2}h+\frac{1}{2}h} - y_{x+\frac{1}{2}h-\frac{1}{2}h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \delta E^{1/2} = \Delta$$

Hence $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(vi) $Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$ [by Taylor's series]

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \cdots$$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)f(x) = e^{hD}f(x)$$

$$\therefore E = e^{hD}$$

$$\text{Cor. } E = 1 + \Delta = e^{hD}$$

Note. A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.

Example : Prove that $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$, the interval of differencing being h.

Solution: Since $\left(\frac{\Delta^2}{E}\right) e^x = \Delta^2 \cdot E^{-1} e^x = \Delta^2 e^{x-h} = \Delta^2 e^x \cdot e^{-h} = e^{-h}\Delta^2 e^x$

$$\therefore \text{R.H.S.} = e^{-h}\Delta^2 e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^{-h} E e^x = e^{-h} \cdot e^{x+h} = e^x$$

Example: Prove with the usual notations, that

- (i) $hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$
- (ii) $\left(E^{1/2} + E^{-1/2}\right)(1 + \Delta)^{1/2} = 2 + \Delta$
- (iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \delta^2/4}$
- (iv) $\Delta^3 y_2 = \nabla^3 y_5$

Solution: *(i)* We know that $e^{hD} = E = 1 + \Delta$

$$\therefore hD = \log(1 + \Delta)$$

$$\text{Also } hD = \log E = -\log(E^{-1}) = -\log(1 - \nabla) \quad [\because E^{-1} = 1 - \nabla]$$

and we have proved that $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

$$\text{i.e. } hD = \sinh^{-1}(\mu\delta)$$

$$\text{Hence } hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$$

(ii) $\left(E^{1/2} + E^{-1/2}\right)(1 + \Delta)^{1/2}$

$$= \left(E^{1/2} + E^{-1/2}\right)E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta$$

(iii) $\frac{1}{2}\delta^2 + \delta\sqrt{1 + \delta^2/4}$

$$= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + (E^{1/2} - E^{-1/2})^2/4}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + \left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)/2$$

$$= \frac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \frac{1}{2}(2E - 2) = E - 1 = \Delta$$

(iv) $\Delta^3 y_2 = (E-1)^3 y_2 \quad [\because \Delta = E - 1]$

$$= (E^3 - 3E^2 + 3E - 1)y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad \ldots(1)$$

$$\nabla^3 y_5 = (1 - E^{-1})^3 y_5 \quad [\because \nabla = 1 - E^{-1}]$$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad \ldots(2)$$

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

Example: Using the method of separation of symbols, prove that

- (i) $u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x}u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \cdots$
- (ii) $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \cdots = e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right)$

Solution: (i) L.H.S. $= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \cdots$

$$= x(1 + xE + x^2E^2 + \cdots)u_1 = x \cdot \frac{1}{1 - xE}u_1, \quad \text{taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1 - x(1+\Delta)}\right]u_1 \quad [\because E = 1 + \Delta]$$

$$= x\left(\frac{1}{1 - x - x\Delta}\right)u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1$$

$$= \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \cdots\right)u_1$$

$$= \frac{x}{1-x}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \cdots = \text{R.H.S.}$$

(ii) L.H.S. $= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2 u_0 + \frac{x^3}{3!}E^3 u_0 + \cdots$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \cdots\right)u_0 = e^{xE}u_0 = e^{x(1+\Delta)}u_0 = e^x \cdot e^{x\Delta}u_0$$

$$= e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \cdots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right) = \text{R.H.S.}$$

Exercises :

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\left(\frac{\Delta^2 u_x}{E u_x}\right)$.

2. Evaluate taking h as the interval of differencing:

   (i) $\frac{\Delta}{E}\sin x$

   (ii) $(\Delta + \nabla)^2(x^2 + x), \quad (h = 1)$

   (iii) $\left(\frac{\Delta^2}{E}\right)x^3$

   (iv) $\frac{\Delta^2 x^3}{E x^3}$

   (v) $\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E \sin(x+h)}$

3. With the usual notations, show that

   (i) $\Delta = 1 - e^{-hD}$

   (ii) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

   (iii) $(1 + \Delta)(1 - \nabla) = 1$

   (iv) $\Delta\nabla = \nabla\Delta = \delta^2$

4. Prove that

   (i) $\delta = \Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2}$

   (ii) $\mu^2 = 1 + \frac{\delta^2}{4}$

   (iii) $\delta\left(E^{1/2} + E^{-1/2}\right) = \Delta E^{-1} + \Delta$

5. Show that

   (i) $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$

   (ii) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

   (iii) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$

6. Show that

   (i) $\Delta = \mu\delta + \frac{\delta^2}{2}$

   (ii) $E^{1/2} = \left(1 + \frac{\delta^2}{4}\right)^{1/2} + \frac{\delta^2}{2}$

   (iii) $E^r = \left(\mu + \frac{1}{2}\delta\right)^{2r}$

7. Prove that

   (i) $\Delta - \nabla = \Delta\nabla$

   (ii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

   (iii) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

(iv) $E = \sum_{i=0}^{\infty} \nabla_i$

(v) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \cdots$

8. Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.
9. Prove with usual notations, that

(i) $\nabla^r f_k = \Delta^r f_{k-r}$

(ii) $\Delta(f_k^2) = (f_k + f_{k+1})\Delta f_k$

Using the method of separation of symbols, prove that

10. $u_0 + u_1 + u_2 + \cdots + u_n = {}^{n+1}\mathrm{C}_1 u_0 + {}^{n+1}\mathrm{C}_2 \Delta u_0 + {}^{n+1}\mathrm{C}_3 \Delta^2 u_0 + \cdots + {}^{n+1}\mathrm{C}_{n+1} \Delta^n u_0$
11. $\Delta^n u_x = u_{x+n} - {}^{n}\mathrm{C}_1 u_{x+n-1} + {}^{n}\mathrm{C}_2 u_{x+n-2} + \cdots + (-1)^n u_x$
12. $y_x = y_n - {}^{n-x}\mathrm{C}_1 \Delta y_{n-1} + {}^{n-x}\mathrm{C}_2 \Delta^2 y_{n-2} - \cdots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$
13. $y_k = \sum_{i=0}^{k} \quad {}^{k}\mathrm{C}_i \Delta^i y_0$

Application to summation of series: The calculus of finite differences is very useful for finding the sum of a given series. The method is best illustrated by the following examples:

Example: Sum the following series $1^3 + 2^3 + 3^3 + \cdots + n^3$.

Solution: Denoting $1^3, 2^3, 3^3$, ... by u_0, u_1, u_2, ... respectively, the required sum

$$S = u_0 + u_1 + u_2 + \cdots + u_{n-1}$$

$$= (1 + E + E^2 + \cdots + E^{n-1})u_0 \quad [\because u_1 = Eu_0, u_2 = E^2 u_0 \text{ etc.}]$$

$$= \frac{E^n - 1}{E - 1} u_0 = \frac{(1 + \Delta)^n - 1}{\Delta} u_0$$

$$= \frac{1}{\Delta}\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \cdots + \Delta^n - 1\right] u_0$$

$$= \left[n + \frac{n(n-1)}{2!}\Delta u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + \cdots\right]$$

Now $\Delta u_0 = u_1 - u_0 = 2^3 - 1^3 = 7$, $\Delta^2 u_0 = u_2 - 2u_1 + u_0 = 3^3 - 2 \cdot 2^3 + 1^3 = 12$,

$$\Delta^3 u_0 = u_3 - 3u_2 + 3u_1 - u_0 = 4^3 - 3 \cdot 3^3 + 3 \cdot 2^3 - 1^3 = 6$$

and $\Delta^4 u_0, \Delta^5 u_0$, ... are all zero as $u_r = r^3$ is a polynomial of third degree.

Hence $S = n + \frac{n(n-1)}{2} \cdot 7 + \frac{n(n-1)(n-2)}{6} \cdot 12 + \frac{n(n-1)(n-2)(n-3)}{24} \cdot 6$

$$= \frac{n^2}{4}(n^2 + 2n + 1) = \left[\frac{n(n+1)}{2}\right]^2$$

Example: Prove that $u_0 + u_1x + u_2x^2 + \cdots \infty = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \cdots \infty$.

Hence find the sum of the series $1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots \infty$.

Solution: $u_0 + u_1x + u_2x^2 + \cdots \infty = (1 + xE + x^2E^2 + x^3E^3 + \cdots \infty)u_0$

$$= \frac{1}{1 - xE}u_0 = \frac{1}{1 - x(1 + \Delta)}u_0$$

$$= \frac{1}{(1 - x) - x\Delta}u_0 = \frac{1}{(1 - x)}\left(1 - \frac{x}{1 - x}\Delta\right)^{-1}u_0$$

$$= \frac{1}{1 - x}\left\{1 - \frac{x\Delta}{1 - x} + \frac{x^2\Delta^2}{(1 - x^2)} + \cdots\right\}u_0$$

$$= \frac{u_0}{1 - x} + \frac{x}{(1 - x)^2}\Delta u_0 + \frac{x^2}{(1 - x)^3}\Delta^2 u_0 + \cdots \infty$$

Now let us construct the difference table for the coefficients of the given series:

u	Δu	Δ^2u	Δ^3u
$u_0 = 2$			
	4		
$u_1 = 6$		2	
	6		0
$u_2 = 12$		2	
	8		0
$u_3 = 20$		2	
	10		
$u_4 = 30$			

This shows that $u_0 = 2, \Delta u_0 = 4, \Delta^2 u_0 = 2, \Delta^3 u_0 = \Delta^4 u_0$ etc. all $= 0$.

Thus $1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots \infty$

$$= u_0 + u_1x + u_2x^2 + \cdots \infty$$

$$= \frac{u_0}{1 - x} + \frac{x}{(1 - x)^2}\Delta u_0 + \frac{x^2}{(1 - x)^3}\Delta^2 u_0 + \cdots \infty$$

$$= \frac{2}{1 - x} + \frac{4x}{(1 - x)^2} + \frac{2x^2}{(1 - x)^3} = \frac{2}{(1 - x)^3}$$

Problems : Using the method of finite differences, sum the following series:

1. $1^2 + 2^2 + 3^2 + \cdots + \mathrm{n}^2$.
2. $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + 11 \cdot 14 + \cdots$ to n terms.
3. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots$ to n terms.
4. $\sum_{\mathrm{x}=1}^{\mathrm{n}} \mathrm{x}\,(\mathrm{x}+2)(\mathrm{x}+4)$.
5. Show that $\mathrm{u}_0 + \frac{\mathrm{u}_1 \mathrm{x}}{1!} + \frac{\mathrm{u}_2 \mathrm{x}^2}{2!} + \cdots = \mathrm{e}^{\mathrm{x}} \left(\mathrm{u}_0 + \frac{\mathrm{x}}{1!} \Delta \mathrm{u}_0 + \frac{\mathrm{x}^2}{2!} \Delta^2 \mathrm{u}_0 + \cdots \right)$.

   Hence sum the series:

   - (i) $1^3 + \frac{2^3}{1!}x + \frac{3^3}{2!}x^2 + \frac{4^3}{3!}x^3 + \cdots \infty$
   - (ii) $1 + \frac{4x}{1!} + \frac{10x^2}{2!} + \frac{20x^3}{3!} + \frac{35x^4}{4!} + \frac{56x^5}{5!} + \cdots \infty$.
6. Find the sum of the series

$$1 \cdot 3 + 3 \cdot 5x + 5 \cdot 7x^2 + 7 \cdot 9x^3 + \cdots \infty.$$

7. Show that $\sum_{r=1}^{n} u_r = {}^{n}\mathrm{C}_1 u_1 + {}^{n}\mathrm{C}_2 \Delta u_1 + {}^{n}\mathrm{C}_3 \Delta^2 u_1 + \cdots + \Delta^{n-1} u_1$.

   Hence evaluate $1^4 + 2^4 + 3^4 + \cdots + n^4$.
8. Sum the series $1 \cdot 2\Delta x^n - 2 \cdot 3\Delta^2 x^n + 3 \cdot 4\Delta^3 x^n - 4 \cdot 5\Delta^4 x^n + \cdots$ to n terms.
9. Show that $\Delta x^n - \frac{1}{2}\Delta^2 x^n + \frac{1 \cdot 3}{2 \cdot 4}\Delta^3 x^n - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\Delta^4 x^n + \cdots$ to n terms $= (x + 1/2)^n - (x - 1/2)^n$.

Interpolation: Suppose we are given the following values of $y = f(x)$ for a set of values of

x:	x_0	x_1	x_2	...	x_n
y:	y_0	y_1	y_2	...	y_n

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called extrapolation. The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications), it is very difficult to determine the exact form of $f(x)$ with the help of tabulated values (x_i, y_i). In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $\phi(x)$ which is known as the interpolating function or smoothing function. If $\phi(x)$ is a polynomial, then it called the interpolating polynomial and the process is called the polynomial interpolation. Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation. But we shall confine ourselves to polynomial interpolation only.

The study of interpolation is based on the calculus of finite differences. We begin by deriving two important interpolation formulae by means of forward and backward differences of a function. These formulae are often employed in engineering and scientific investigations.

Newton's Forward Interpolation Formula: Let the function $y = f(x)$ take the values $y_0, y_1, \dots, y_n$ corresponding to the values $x_0, x_1, \dots, x_n$ of x. Let these values of x be equi-spaced such that $x_i = x_0 + ih$ $(i = 0,1, \dots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \dots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \quad \dots (1)$$

Putting $x = x_0, x_1, \dots, x_n$ successively in (1), we get

$$y_0 = a_0, \quad y_1 = a_0 + a_1(x_1 - x_0), \quad y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore a_1 = \frac{1}{h}\Delta y_0$$

Also $\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1) = a_1 h + a_2 \cdot 2h \cdot h = \Delta y_0 + 2h^2 a_2$

$$\therefore a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2}\Delta^2 y_0$$

Similarly $a_3 = \frac{1}{3!h^3}\Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \quad \dots (2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph, \quad x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = (p - 1)h,$$

$$x - x_2 = x - x_1 - (x_2 - x_1) = (p - 1)h - h = (p - 2)h \quad \text{etc.}$$

Hence, writing $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1) \dots (p-n+1)}{n!}\Delta^n y_0 \quad \dots (3)$$

It is called Newton's forward interpolation formula as (3) contains y_0 and the forward differences of y_0.

Otherwise: Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

Thus $y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0 \quad [\because E = 1 + \Delta]$

Interpolation

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right\} y_0 \quad \ldots (4)$$

i.e.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

(using Binomial theorem)

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1} y_0$ and higher differences will be zero. Hence (4) will become $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-n-1)}{n!}\Delta^n y_0$ which is same as (3).

This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.

Newton's Backward Interpolation Formula: Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \quad [\because E^{-1} = 1 - \nabla$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\right] y_n$$

(using Binomial theorem)

i.e.

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \quad \ldots (1)$$

It is called Newton's backward interpolation formula as (1) contains y_n and backward differences of y_n.

This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of y_n.

Example . The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x=height:	100	150	200	250	300	350	400
$y = distance$:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(i) $x = 218$ ft

(ii) $x = 410$

Solution: The difference table is as under :

x	y	Δy	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		−0.39		
		2.01		0.15	
200	15.04		−0.24		−0.07
		1.77		0.08	
250	16.81		−0.16		−0.05
		1.61		0.03	
300	18.42		−0.13		−0.01
		1.48		0.02	
350	19.90		−0.11		
		1.37			
400	21.27				

(i) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 = 0.03$ etc.

Since $x = 218$ and $h = 50$, ∴ $p = \frac{x - x_0}{h} = \frac{18}{50} = 0.36$

∴ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{0.36(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

$$= 15.697\ i.e.\ 15.7\ nautical\ miles.$$

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

∴ Taking $x_n = 400, p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

∴ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400}$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53\ nautical\ miles.$$

Example: From the following table, estimate the number of students who obtained marks between 40 and 45 :

Marks :	30–40	40–50	50–60	60–70	70–80
No. of students :	31	42	51	35	31

Sol. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \quad [\because h = 10$$

∴ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \ on\ simplification.$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and $45 = 48 - 31 = 17$.

Example : Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

2. From the following table, find y when $x = 1.85$ and 2.4 by Newton's interpolation formulae.

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

Central Difference Interpolation Formulae: In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (= \delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

Example: From the following table, estimate the number of students who obtained marks between 40 and 45 :

Marks :	30–40	40–50	50–60	60–70	70–80
No. of students :	31	42	51	35	31

Solution: First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40, x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \quad [\because h = 10$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25)$$
$$+ \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$
$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$
$$= 47.87, \ on\ simplification.$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and $45 = 48 - 31 = 17$.

Example: Find the cubic polynomial which takes the following values :

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate $f(4)$.

Solution: The difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \frac{x-0}{h} = x$ [$\because h = 1$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$
$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$
$$= 2x^3 - 7x^2 + 6x + 1,$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3, x = 4$ so that $p = \frac{x - x_n}{h} = 1 \quad [\because h = 1$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.

Example: In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution: The difference table is

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	Δ^5y
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1$, $h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9$, $x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

Problems :1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

2. From the following table, find y when $x = 1.85$ and 2.4 by Newton's interpolation formulae :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. If $f(1.15) = 1.0723, f(1.20) = 1.0954, f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.

4. Given $\sin 45° = 0.7071, \sin 50° = 0.7660, \sin 55° = 0.8192, \sin 60° = 0.8660$, find $\sin 52°$ using Newton's forward formula.

5. From the following table :

x :	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately.

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

7. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

Hence evaluate y for $x = 5$

8. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x :

x :	1	2	3	4	5
y :	1	-1	1	-1	1

9. Given $u_1 = 40, u_3 = 45, u_5 = 54$, find u_2 and u_4.

10. If $u_{-1} = 10, u_1 = 8, u_2 = 10, u_4 = 50$, find u_0 and u_3.

11. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

12. Given $\sum_1^{10} f(x) = 500426, \sum_4^{10} f(x) = 329240, \sum_7^{10} f(x) = 175212$ and $f(10) = 40365$, find $f(1)$

Interpolation with Unequal Intervals :

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(i) Lagrange's interpolation formula

(ii) Newton's general interpolation formula with divided differences. LAGRANGE'S **Interpolation Formula:**

If $y = f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$, then

$$f(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} y_1 + \cdots + \frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})} y_n$$

This is known as Lagrange's interpolation formula for unequal intervals.

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots (x - x_n)$$

$$+a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)+\cdots+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

Putting $x=x_0, y=y_0$, in (2), we get

$$y_0=a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$
$$a_0=y_0/[(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)]$$

Similarly putting $x=x_1, y=y_1$ in (2), we have $a_1=y_1/[(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)]$

Proceeding the same way, we find $a_2, a_3 \ldots a_n$.

Substituting the values of $a_0, a_1, \ldots, a_n$ in (2), we get (1).

Obs. 1. Lagrange's formula can be applied whether the values x_i are equally spaced or not. It is easy to remember but quite cumbersome to apply.

Obs. 2. This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x-x_0)(x-x_1)\ldots(x-x_n)$, we get

$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots(x-x_n)}=\frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\cdot\frac{1}{x-x_0}+\frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\cdot\frac{1}{x-x_1}+\cdots+\frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\cdot\frac{1}{x-x_n}$$

Divided Differences: The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0,y_0),(x_1,y_1),(x_2,y_2),\ldots$ be given points, then the first divided difference for the arguments x_0, x_1 is defined by the relation $[x_0,x_1]=\frac{y_1-y_0}{x_1-x_0}$

Similarly $[x_1,x_2]=\frac{y_2-y_1}{x_2-x_1}$ and $[x_2,x_3]=\frac{y_3-y_2}{x_3-x_2}$ etc.

The second divided difference for x_0, x_1, x_2 is defined as $[x_0,x_1,x_2]=\frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0,x_1,x_2,x_3]=\frac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_3-x_0}\quad\text{and so on.}$$

Obs. 1. The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments. For it is easy to write $[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} = [x_1, x_0]$, $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. The nth divided differences of a polynomial of the nth degree are constant.

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \frac{1}{2!\,h^2}\Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots, x_n] = \frac{1}{n!\,h^n}\Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

Newton's Divided Difference Formula:

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that

$$y = y_0 + (x - x_0)[x, x_0]$$

Again

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$

Also

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots + (x - x_0)(x - x_1) \ldots (x - x_n)[x, x_0, x_1, \ldots, x_n]$$

which is called Newton's general interpolation formula with divided differences.

Example : Given the values

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate $f(9)$, using
(i) Lagrange's formula
(ii) Newton's divided difference formula.

Solution: (i) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$
and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$\begin{aligned} f(9) = {} & \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\ & + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\ & + \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\ & + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\ & + \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\ = {} & -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810 \end{aligned}$$

(ii) The divided differences table is

x	y	1st divided differences	2nd divided differences	3rd divided differences
5	150			
		121		
7	392		24	
		265		1
11	1452		32	
		457		1
13	2366		42	
		709		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example: Determine f(x) as a polynomial in x for the following data :

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335.

Solution: The divided differences table is

x	f(x)	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		

x	f(x)	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$
$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+(x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Problems :

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16.

2. The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of $140°$.

Temp.∘ :	110	130	160	190
Viscosity :	10.8	8.1	5.5	4.8

3. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. *(Nagarjuna B.E., 2001)*

4. Given $y_0 = -12$, $y_1 = 0$, $y_3 = 6$ and $y_4 = 12$, find y_2.

5. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792.

6. If $y(1) = -3, y(3) = 9, y(4) = 30, y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points.

7. Given $f(0) = -18, f(1) = 0, f(3) = 0, f(5) = -248, f(6) = 0, f(9) = 13104$, find $f(x)$.

8. Using Lagrange's formula, express the function $\frac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions.

[Hint. Tabulate the values of $f(x) = x^2 + 6x - 1$ for $x = -1,1,4,6$ and apply Lagrange's formula.]

9. Certain corresponding values of x and $\log_{10} x$ are given below :

x :	300	304	305	307
$\log_{10} x$:	2.4771	2.4829	2.4843	2.4871

Find $\log_{10} 310$ by

(i) Lagrange's formula.

(ii) Newton's divided difference formula.

10. Use Newton's divided difference formula to find $f(x)$ from the following data :

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

11. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Lagrange's Method: This procedure is similar to Lagrange's interpolation formula only difference being that x is assumed to be expressible as a polynomial in y. Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}x_1$$
$$+\dots+\frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})}\dots$$

which is used for inverse interpolation.

Example . The following table gives the values of x and y :

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique.

Solution: Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6.$

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.3)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$
$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$
$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Iterative Method: Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

From this, we get

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \cdots\right]$$

Neglecting the second and higher differences, we obtain the first approximation to p as

$$p_1 = (y_p - y_0)/\Delta y_0$$

To find the second approximation, retaining the term with second differences in (1) and replacing p by p_1, we get

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right]$$

To find the third approximation, retaining the term with third differences in (1) and replacing every p by p_2, we have

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0\right]$$

and so on. This process is continued till two successive approximations of p agree with each other.

This method is a powerful iterative procedure for finding the roots of an equation to a good degree of accuracy.

Example: Using inverse interpolation, find the real root of the equation $x^3 + x - 3 = 0$, which is close to 1.2.

Solution: The difference table is

x	v	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	-0.2	-1				
			0.431			
1.1	-0.1	-0.569		0.066		
			0.497		0.006	
1.2	0	-0.072		0.072		0
			0.569		0.006	
1.3	0.1	0.497		0.078		
			0.647			
1.4	0.2	1.144				

Clearly the root of the given equation lies between 1.2 and 1.3. Assuming the origin at $x = 1.2$ and using Stirling's formula.

$$y = y_0 + x\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{x^2}{2}\Delta^2 y_{-1} + \frac{x(x^2-1)}{6}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}, \text{ we get}$$

$$0 = -0.072 + x \cdot \frac{0.569 + 0.467}{2} + \frac{x^2}{2}(0.072) + \frac{x(x^2-1)}{6}\frac{0.006 + 0.006}{2} \quad (\because y = 0)$$

$$\text{or } 0 = -0.072 + 0.532x + 0.036x^2 + 0.001x^3$$

This equation can be written as

$$x = \frac{0.072}{0.532} - \frac{0.036}{0.532}x^2 - \frac{0.001}{0.532}x^3$$

$$\therefore \text{ First approximation } x^{(1)} = \frac{0.072}{0.532} = 0.1353$$

Putting $x = x^{(1)}$ on R.H.S. of (i), we get

Second approximation $x^{(2)} = 0.1353 - 0.067(0.1353)^2 - 1.8797(0.1353)^3 = 0.134$

Hence the desired root $= 1.2 + 0.1 \times 0.134 = 1.2134$.

Problems :

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

3. Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$, given that $f(30) = -30, f(34) = -13, f(38) = 3$ and $f(42) = 18$.

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